# EC351: Answers to Problem Set 1

1.

(a)
$$f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{3}}$$

Partial derivatives are:

$$f_1 = \frac{x_1^{-\frac{1}{2}} x_2^{\frac{1}{3}}}{2}$$
$$f_2 = \frac{x_1^{\frac{1}{2}} x_2^{-\frac{2}{3}}}{3}$$
$$f_{11} = -\frac{x_1^{-\frac{3}{2}} x_2^{\frac{1}{3}}}{4}$$
$$f_{22} = -\frac{2x_1^{\frac{1}{2}} x_2^{-\frac{5}{3}}}{9}$$
$$f_{21} = f_{12} = \frac{x_1^{-\frac{1}{2}} x_2^{-\frac{2}{3}}}{6}$$

Construct the Hessian matrix and find the signs of the leading principal minors

$$\mid H \mid = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} -\frac{x_1^{-\frac{3}{2}} x_2^{\frac{1}{3}}}{4} & \frac{x_1^{-\frac{1}{2}} x_2^{-\frac{2}{3}}}{6} \\ \frac{x_1^{-\frac{1}{2}} x_2^{-\frac{2}{3}}}{6} & -\frac{2x_1^{\frac{1}{2}} x_2^{-\frac{5}{3}}}{9} \end{vmatrix}$$

$$\mid H_1 \mid = -\frac{x_1^{-\frac{3}{2}} x_2^{\frac{1}{3}}}{4} < 0$$

$$\mid H_2 \mid = \frac{2x_1^{-1} x_2^{-\frac{4}{3}}}{36} - \frac{x_1^{-1} x_2^{-\frac{4}{3}}}{36}$$
$$= \frac{x_1^{-1} x_2^{-\frac{4}{3}}}{36} > 0$$

Therefore $\mid H \mid$ is negative definite and so $f$ is strictly concave for any $x_1, x_2 > 0$.

(b) $$f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$$

Partial derivatives are:

$$f_1 = \frac{x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}}}{2}$$
$$f_2 = \frac{x_1^{\frac{1}{2}} x_2^{-\frac{1}{2}}}{2}$$
$$f_{11} = -\frac{x_1^{-\frac{3}{2}} x_2^{\frac{1}{2}}}{4}$$
$$f_{22} = -\frac{x_1^{\frac{1}{2}} x_2^{-\frac{3}{2}}}{4}$$
$$f_{21} = f_{12} = \frac{x_1^{-\frac{1}{2}} x_2^{-\frac{1}{2}}}{4}$$

Construct the Hessian matrix and find the signs of the leading principal minors

$$| H_1 | = -\frac{x_1^{-\frac{3}{2}} x_2^{\frac{1}{2}}}{4} < 0$$

$$| H_2 | = \frac{x_1^{-1} x_2^{-1}}{8} - \frac{x_1^{-1} x_2^{-1}}{8} = 0$$

Therefore $f$ is not strictly concave. Check for (weak) concavity.

$$| H_1^* | = f_{11}, f_{22} < 0$$

$$| H_2^* | = | H_2 | \geq 0$$

Therefore $f$ is concave.

(c) $$f(x_1,x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{4}}$$

Partial derivatives are:

$$f_1 = \frac{x_1^{-\frac{1}{2}} x_2^{\frac{1}{4}}}{2}$$
$$f_2 = \frac{x_1^{\frac{1}{2}} x_2^{-\frac{3}{4}}}{4}$$
$$f_{11} = -\frac{x_1^{-\frac{3}{2}} x_2^{\frac{1}{4}}}{4}$$
$$f_{22} = -\frac{3x_1^{\frac{1}{2}} x_2^{-\frac{7}{4}}}{16}$$
$$f_{21} = f_{12} = \frac{x_1^{-\frac{1}{2}} x_2^{-\frac{3}{4}}}{8}$$

Construct the Bordered Hessian and find the signs of the leading principal minors

$$\mid \overline{H} \mid = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} 0 & \frac{x_1^{-\frac{1}{2}} x_2^{\frac{1}{4}}}{2} & \frac{x_1^{\frac{1}{2}} x_2^{-\frac{3}{4}}}{4} \\ \frac{x_1^{-\frac{1}{2}} x_2^{\frac{1}{4}}}{2} & -\frac{x_1^{-\frac{3}{2}} x_2^{\frac{1}{4}}}{4} & \frac{x_1^{-\frac{1}{2}} x_2^{-\frac{3}{4}}}{8} \\ \frac{x_1^{\frac{1}{2}} x_2^{-\frac{3}{4}}}{4} & \frac{x_1^{-\frac{1}{2}} x_2^{-\frac{3}{4}}}{8} & -\frac{3x_1^{\frac{1}{2}} x_2^{-\frac{7}{4}}}{16} \end{vmatrix}$$

$$\mid \overline{H}_1 \mid = -\frac{x_1^{-1} x_2^{\frac{1}{2}}}{4} < 0$$

$$\mid \overline{H}_2 \mid = \frac{3x_1^{-\frac{1}{2}} x_2^{-\frac{5}{4}}}{32} > 0$$

Therefore $f$ is strictly quasiconcave for any $x_1, x_2 > 0$. Is $f$ also strictly concave?

$$f_{11} < 0$$

$$f_{11}f_{22} > f_{12}^2 = \frac{3x_1^{-1} x_2^{-\frac{3}{2}}}{64} > \frac{x_1^{-1} x_2^{-\frac{3}{2}}}{64} \Rightarrow > 0$$

Thus $f$ is also strictly concave.

(d) $$f(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$$

Partial derivatives are:

$$\begin{aligned}
f_1 &= \frac{x_1^{-\frac{2}{3}} x_2^{\frac{2}{3}}}{3} \\
f_2 &= \frac{2x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}}}{3} \\
f_{11} &= -\frac{2x_1^{-\frac{5}{3}} x_2^{\frac{2}{3}}}{9} \\
f_{22} &= -\frac{2x_1^{\frac{1}{3}} x_2^{-\frac{4}{4}}}{9} \\
f_{21} = f_{12} &= \frac{2x_1^{-\frac{2}{3}} x_2^{-\frac{1}{3}}}{9}
\end{aligned}$$

Construct the Bordered Hessian and find the signs of the leading principal minors

$$| \overline{H} | = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} 0 & \frac{x_1^{-\frac{2}{3}} x_2^{\frac{2}{3}}}{3} & \frac{2x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}}}{3} \\ \frac{x_1^{-\frac{2}{3}} x_2^{\frac{2}{3}}}{3} & -\frac{2x_1^{-\frac{5}{3}} x_2^{\frac{2}{3}}}{9} & \frac{2x_1^{-\frac{2}{3}} x_2^{-\frac{1}{3}}}{9} \\ \frac{2x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}}}{3} & \frac{2x_1^{-\frac{2}{3}} x_2^{-\frac{1}{3}}}{9} & -\frac{2x_1^{\frac{1}{3}} x_2^{-\frac{4}{3}}}{9} \end{vmatrix}$$

$$| \overline{H}_1 | = -\frac{x_1^{-\frac{4}{3}} x_2^{\frac{4}{3}}}{9} < 0$$

$$| \overline{H}_2 | = \frac{18x_1^{-1}}{81} > 0$$

Therefore $f$ is strictly quasiconcave for any $x_1, x_2 > 0$. Is $f$ also strictly concave?

$$\begin{aligned}
f_{11} &< 0 \\
f_{11} f_{22} &= f_{12}^2
\end{aligned}$$

Thus $f$ is not strictly concave since $| H_2 | = 0$. However $f$ is (weakly) concave since:

$$\begin{aligned}
| H_1^* | &= f_{11}, f_{22} < 0 \\
| H_2^* | &= | H_2 | \geq 0
\end{aligned}$$

(e) $$f(x_1, x_2) = 3x_1^4 + 5x_2^2$$

Partial derivatives are:

$$\begin{aligned} f_1 &= 12x_1^3 \\ f_2 &= 10x_2 \\ f_{11} &= 36x_1^2 \\ f_{22} &= 10 \\ f_{21} = f_{12} &= 0 \end{aligned}$$

Construct the Bordered Hessian and find the signs of the leading principal minors

$$| \overline{H} |= \begin{vmatrix} 0 & 12x_1^3 & 10x_2 \\ 12x_1^3 & 36x_1^2 & 0 \\ 10x_2 & 0 & 10 \end{vmatrix}$$

$$| \overline{H}_1 |= -144x_1^6 < 0$$

$$| \overline{H}_2 |= -1440x_1^6 - 3600x_1^2x_2^2 < 0$$

Therefore $f$ is strictly quasiconvex for any $x_1, x_2 > 0$. Is $f$ also strictly convex?

$$\begin{aligned} f_{11} &= 36x_1^2 > 0 \\ f_{11}f_{22} > f_{12}^2 &= 360x_1^2 > 0 \end{aligned}$$

Thus $f$ is also strictly convex.

2.

$$\Pi = [100 - (Q_1 + Q_2)] (Q_1 + Q_2) - 2Q_1^2 - 3Q_2^2$$

<u>Nec:</u>

$$\Pi_1 = 100 - 6Q_1 - 2Q_2 = 0$$
$$\Pi_2 = 100 - 2Q_1 - 8Q_2 = 0$$

Solving for $Q_1^*$ and $Q_2^*$ yields:

$$Q_1^* = \frac{150}{11}$$
$$Q_2^* = \frac{100}{11}$$

<u>Suff:</u> Construct the Hessian matrix and find the signs of the leading principal minors

$$| H |= \begin{vmatrix} -6 & -2 \\ -2 & -8 \end{vmatrix}$$

$$| H_1 |= -6 < 0$$

$$| H_2 |= 48 - 4 = 44 > 0$$

Local maximum at $Q_1^* = \frac{150}{11}$ and $Q_2^* = \frac{100}{11}$. Note that since the profit function is also a strictly concave function for every $Q_1$, $Q_2$ this is also a global maximum.

3.

$$f = x^{\alpha} y^{\beta}$$

Partial derivatives are:

$$\begin{aligned} f_1 &= \alpha x^{\alpha-1} y^{\beta} \\ f_2 &= \beta x^{\alpha} y^{\beta-1} \\ f_{11} &= \alpha(\alpha-1) x^{\alpha-2} y^{\beta} \\ f_{22} &= \beta(\beta-1) x^{\alpha} y^{\beta-2} \\ f_{21} = f_{12} &= \alpha\beta x^{\alpha-1} y^{\beta-1} \end{aligned}$$

Construct the Hessian matrix and find the signs of the leading principal minors

$$\mid H \mid = \begin{vmatrix} \alpha(\alpha-1)x^{\alpha-2}y^{\beta} & \alpha\beta x^{\alpha-1}y^{\beta-1} \\ \alpha\beta x^{\alpha-1}y^{\beta-1} & \beta(\beta-1)x^{\alpha}y^{\beta-2} \end{vmatrix}$$

$$\mid H_1 \mid = \alpha(\alpha-1)x^{\alpha-2}y^{\beta} < 0 \Leftrightarrow \alpha < 1$$

$$\begin{aligned} \mid H_2 \mid = \alpha\beta(\alpha-1)(\beta-1)x^{2\alpha-2}y^{2\beta-2} &> \alpha^2\beta^2 x^{2\alpha-2}y^{2\beta-2} \\ &= (\alpha-1)(\beta-1) > \alpha\beta \\ &= \alpha+\beta < 1 \end{aligned}$$

for strict concavity.

To test for quasiconcavity, construct the Bordered Hessian and find the signs of the leading principal minors

$$\mid H \mid = \begin{vmatrix} 0 & \alpha x^{\alpha-1}y^{\beta} & \beta x^{\alpha}y^{\beta-1} \\ \alpha x^{\alpha-1}y^{\beta} & \alpha(\alpha-1)x^{\alpha-2}y^{\beta} & \alpha\beta x^{\alpha-1}y^{\beta-1} \\ \beta x^{\alpha}y^{\beta-1} & \alpha\beta x^{\alpha-1}y^{\beta-1} & \beta(\beta-1)x^{\alpha}y^{\beta-2} \end{vmatrix}$$

$$\mid \overline{H}_1 \mid = -\alpha^2 x^{2\alpha-2} y^{2\beta} < 0$$

$$\mid \overline{H}_2 \mid = x^{3\alpha-2} y^{3\beta-2} \left[\beta\alpha^2 + \beta^2\alpha\right] > 0$$

for all $\alpha, \beta > 0$.

4.

$$\Theta = xz + x^2 - y + yz + y^2 + 3z^2$$

<u>Nec:</u>

$$\Theta_x = z + 2x = 0$$
$$\Theta_y = -1 + z + 2y = 0$$
$$\Theta_z = x + y + 6z = 0$$

Solving for $x^*, y^*$ and $z^*$ yields:

$$x^* = \frac{1}{20}$$
$$y^* = \frac{11}{20}$$
$$z^* = -\frac{1}{10}$$

<u>Suff:</u> Construct the Hessian matrix and find the signs of the leading principal minors

$$\mid H \mid = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{vmatrix}$$

$$\mid H_1 \mid = 2 > 0$$

$$\mid H_2 \mid = 4 > 0$$
$$\mid H_3 \mid = 20 > 0$$

Hence the function is strictly convex for every $x$,$y$ and $z$ and thus we have a global minimum.