EC351: Answers to Problem Set 2

1)

a)

The Lagrangian function is:

$$L = 2x + 3y + \lambda[10 - 2x^2 - 5y^2]$$

First-order conditions:

$$L_x = 2 - 4\lambda x = 0$$

$$L_y = 3 - 10\lambda y = 0$$

$$L_\lambda = 10 - 2x^2 - 5y^2 = 0$$

Solve for x^*,y^*,λ^*:

$$x^* = \sqrt{\frac{50}{19}}$$
$$y^* = \frac{3}{5}\sqrt{\frac{50}{19}}$$
$$\lambda^* = \frac{2}{4x^*}$$

Second-order conditions:

$$g_x = 4x$$
$$g_y = 10y$$
$$L_{xx} = -4\lambda$$
$$L_{yy} = -10\lambda$$
$$L_{xy} = L_{yx} = 0$$

Derive the Bordered Hessian matrix of the Lagrange function:

$$H^B = \begin{bmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{bmatrix} = \begin{bmatrix} 0 & 4x^* & 10y^* \\ 4x^* & -4\lambda^* & 0 \\ 10y^* & 0 & -10\lambda^* \end{bmatrix}$$

Check the sign of the leading principal minors

$$\begin{aligned} \mid H_1^B \mid &= -16x^{*2} < 0 \\ \mid H_2^B \mid &= -4x^*[-40x^*\lambda^*] + 10y^*(40y^*\lambda^*) \\ &= 160x^{*2}\lambda^* + 400y^{*2}\lambda^* \\ &= \lambda^*[160x^{*2} + 400y^{*2}] > 0 \end{aligned}$$

since λ^*,x^* and $y^* > 0$.

Local maximum at x^* and y^*.

Is the solution a global maximum?

Objective function is linear $\Rightarrow$ quasiconcave.

Constraint? Construct a bordered Hessian to test for quasiconvexity where x^* and $y^* > 0$.

$$\overline{H} = \begin{bmatrix} 0 & 4x^* & 10y^* \\ 4x^* & 4 & 0 \\ 10y^* & 0 & 10 \end{bmatrix}$$

$$
\begin{aligned}
| \overline{H}_1 | &= -16x^{*2} < 0 \\
| \overline{H}_2 | &= -4x^*[-40x^*] + 10y^*(-40y^*) \\
&= 160x^{*2} - 400y^{*2} < 0
\end{aligned}
$$

Thus the constraint is a strictly quasiconvex function so we have a unique global maximum.

b)

$$L = (x+2)(y+1) + \lambda[m - x - y]$$

First-order conditions:

$$L_x = y + 1 - \lambda = 0$$

$$L_y = x + 2 - \lambda = 0$$

$$L_\lambda = m - x - y = 0$$

Solve for x^*,y^*,λ^*:

$$
\begin{aligned}
x^* &= \frac{m-1}{2} \\
y^* &= \frac{m+1}{2} \\
\lambda^* &= \frac{m+3}{2}
\end{aligned}
$$

Second-order conditions:

$$g_x = 1$$
$$g_y = 1$$
$$L_{xx} = 0$$
$$L_{yy} = 0$$
$$L_{xy} = L_{yx} = 1$$

Derive the Bordered Hessian matrix of the Lagrange function:

$$H^B = \begin{bmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Check the sign of the leading principal minors

$$\mid H_1^B \mid = -1 < 0$$
$$\mid H_2^B \mid = -1(-1) + 1(1)$$
$$= 1 + 1 = 2 > 0$$

Local maximum at x^* and y^*.

Is the solution a global maximum?

Constraint is linear $\Rightarrow$ quasiconvex.

Objective function? Construct a bordered Hessian to test for quasiconcavity where x^* and $y^* > 0$. [This requires $m > 1$ otherwise $x^* < 0$.

$$\overline{H} = \begin{bmatrix} 0 & y^* + 1 & x^* + 2 \\ y^* + 1 & 0 & 1 \\ x^* + 2 & 1 & 0 \end{bmatrix}$$

$$| \overline{H}_1 | = -(y^* + 1)^2 < 0$$
$$| \overline{H}_2 | = (y^* + 1)(x^* + 2) + (x^* + 2)(y^* + 1) > 0$$

Thus objective function is strictly quasiconcave so we have a unique global maximum.

2)

$$L = 4xyz^2 + \lambda[56 - x - y - z]$$

First-order conditions:

$$L_x = 4yz^2 - \lambda = 0$$

$$L_y = 4xz^2 - \lambda = 0$$

$$L_z = 8xyz - \lambda = 0$$

$$L_\lambda = 56 - x - y - z = 0$$

Solve for x^*,y^*,z^*,λ^*:

$$x^* = 14$$
$$y^* = 14$$
$$z^* = 28$$
$$\lambda^* = 43904$$

Second-order conditions:

$$g_x = 1$$
$$g_y = 1$$
$$g_z = 1$$
$$L_{xx} = 0$$
$$L_{yy} = 0$$
$$L_{zz} = 8xy$$
$$L_{xy} = L_{yx} = 4z^2$$
$$L_{xz} = L_{zx} = 8yz$$
$$L_{yz} = L_{zy} = 8xy$$

Derive the Bordered Hessian matrix of the Lagrange function:

$$H^B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 4z^{*2} & 8y^*z^* \\ 1 & 4z^{*2} & 0 & 8x^*z^* \\ 1 & 8y^*z^* & 8x^*z^* & 8x^*y^* \end{bmatrix}$$

Check the sign of the leading principal minors

$$\mid H_1^B \mid = -1 < 0$$
$$\mid H_2^B \mid = -1(-4z^{*2}) + 1(4z^{*2}) = 8z^* > 0$$

$$| H_3^B |= -1 \begin{vmatrix} 1 & 4z^{*2} & 8y^*z^* \\ 1 & 0 & 8x^*z^* \\ 1 & 8x^*z^* & 8x^*y^* \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 & 8y^*z^* \\ 1 & 4z^{*2} & 8x^*z^* \\ 1 & 8y^*z^* & 8x^*y^* \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 4z^{*2} \\ 1 & 4z^{*2} & 0 \\ 1 & 8y^*z^* & 8x^*z^* \end{vmatrix}$$

$$| H_3^B |= 16z^{*4} - 64x^*z^{*3} - 64y^*z^{*3} - 64x^*y^*z^{*2} + 64x^{*2}z^{*2} + 64y^{*2}z^{*2}$$

Evaluated at $x^* = 14$, $y^* = 14$, $z^* = 28$:

$| H_3^B |= -19,668,992 < 0.$

Local maximum at $x^* = 14$, $y^* = 14$, $z^* = 28$.

Substitute $x^* = 14$, $y^* = 14$, $z^* = 28$ and $\lambda = 43904$ into the Lagrangean:

$$L_0 = 4(14)(14)(28)^2 + 43904[56 - 14 - 14 - 28] = 614656$$
$$L_1 = L_0 + \lambda \approx 614,656 + 43,904 \approx 658,560.$$

3)

$$L = (x - a)(y - b) + \lambda(m - px - qy)$$

F.O.C.s

$$L_x = y - b - \lambda p = 0$$

$$L_y = x - a - \lambda q = 0$$

$$L_\lambda = m - px - qy = 0$$

Solve for x^*,y^*,λ^*:

$$x^* = (m - bq + pa)/2p$$
$$y^* = ((m - bq - pq)/2q) + b$$
$$\lambda^* = (m - bq - pa)/2pq$$

Construct the value function V:

$$V = (x^* - a)(y^* - b) = \left(\frac{m - bq + pa}{2p} - a\right)\left(\frac{m - bq - pa}{2q}\right)$$

$$\frac{\partial V}{\partial m} = \frac{m - bq - pa}{2pq} = \lambda^*$$