

EC351: Answers to Problem Set 3

1.a) Find the comparative-static effect of a change in α :

$$\max z = x^{0.25} y^{0.75} \text{ subject to } 2x + 4y = \alpha$$

$$L = x^{0.25} y^{0.75} + \lambda[\alpha - 2x - 4y]$$

First-order conditions (FOCs):

$$L_x = 0.25x^{-0.75}y^{0.75} - 2\lambda = 0$$

$$L_y = 0.75x^{0.25}y^{-0.25} - 4\lambda = 0$$

$$L_\lambda = \alpha - 2x - 4y = 0$$

Eliminate λ by dividing the first equation and the second:

$$\frac{0.25 x^{-0.75} y^{0.75}}{0.75 x^{0.25} y^{-0.25}} = \frac{1}{2}$$

Rearranging that expression:

$$\frac{y^{0.25} y^{0.75}}{x^{0.25} x^{0.75}} = \frac{0.75}{0.5}$$

$$\Rightarrow \frac{y}{x} = 1.5$$

Using the constraint, solve for x^* and y^* :

$$x^* = \frac{\alpha}{8}$$

$$y^* = \frac{3\alpha}{16}$$

The value function is given by:

$$V = \left(\frac{\alpha}{8}\right)^{0.25} \left(\frac{3\alpha}{16}\right)^{0.75} = 0.17\alpha$$

Therefore:

$$\frac{\partial V}{\partial \alpha} = 0.17 > 0$$

Obviously we could have used the fact that:

$$\frac{\partial V}{\partial \alpha} = \lambda^*$$

Find λ^* from the first FOC:

$$\lambda = \frac{0.25 (x^*)^{-0.75} (y^*)^{0.75}}{2}$$
$$\Rightarrow \frac{0.25 \left(\frac{3\alpha}{16}\right)^{0.75}}{2 \left(\frac{\alpha}{8}\right)^{0.75}}$$

$$\lambda^* = 0.125 \left(\frac{3\alpha}{16} \frac{8}{\alpha}\right)^{0.75} = 0.17$$

1.b) Find the comparative-static effect of a change in α :

$$\min z = 2x + 4y \quad \text{subject to} \quad x^{0.25} y^{0.75} = \alpha$$

This is the DUAL problem of exercise 1.a). For example, if 1.a) was the maximization of the utility function given a budget constraint, exercise 1.b) is the minimization of the expenditure function given a certain level of utility (α).

$$L = 2x + 4y + \lambda[\alpha - x^{0.25} y^{0.75}]$$

First-order conditions:

$$L_x = 2 - 0.25\lambda x^{-0.75}y^{0.75} = 0$$

$$L_y = 4 - 0.75\lambda x^{0.25}y^{-0.25} = 0$$

$$L_\lambda = \alpha - x^{0.25}y^{0.75} = 0$$

Proceed as in exercise 1.a):

$$x^* = \alpha(1.5)^{-0.75}$$

$$y^* = \alpha(1.5)^{0.25}$$

The Value Function V:

$$V = 2\alpha(1.5)^{-0.75} + 4\alpha(1.5)^{0.25} = \alpha 5.91$$

Therefore:

$$\frac{\partial V}{\partial \alpha} = 5.91 > 0$$

As before you can find the same effect by calculating λ^*

2) Suppose a monopolist uses L workers to produce Q units of output:

$$\Pi(Q, L) = Q(a - bQ) - wL$$

where a, b are positive constants determining demand and w is the cost of labour. Given the production function $Q = 2L^{0.5}$:

- a) Find the firm's optimal employment decision L^*
- b) How does employment change as a and w change?

$$L = aQ - bQ^2 - wL + \lambda[2L^{0.5} - Q]$$

First-order conditions:

$$L_Q = a - 2bQ - \lambda = 0$$

$$L_L = -w + \lambda L^{-0.5} = 0$$

$$L_\lambda = 2L^{0.5} - Q = 0$$

To eliminate λ : rewrite the first two equations as:

$$\lambda = a - 2bQ$$

$$\lambda = \frac{w}{L^{-0.5}}$$

Thus equating the right-hand side of those two equations:

$$a - 2bQ = \frac{w}{L^{-0.5}}$$

$$\Rightarrow Q = \frac{a}{2b} - \frac{wL^{0.5}}{2b}$$

Using that and the constraint, you can solve for Q^* and L^* :

From the constraint we have: $L^{0.5} = \frac{Q}{2}$.

Therefore: $Q = \frac{a}{2b} - \frac{wQ}{4b}$

$$Q + \frac{wQ}{4b} = \frac{a}{2b}$$

$$\Rightarrow Q \left(\frac{4b + w}{4b} \right) = \frac{a}{2b}$$

$$\Rightarrow Q = \frac{2a}{4b + w}$$

Using the last result into the constraint:

$$L^{0.5} = \frac{2a}{4b + w} \frac{1}{2}$$

$$\Rightarrow L = \left[\frac{2a}{4b + w} \right]^2$$

Therefore our solutions are:

$$Q^* = \frac{2a}{w + 4b}$$

$$L^* = \left[\frac{a}{w + 4b} \right]^2$$

How does employment change as a and w change?

$$\frac{\partial L^*}{\partial a} = \frac{2a}{(w + 4b)^2} > 0$$

Meaning: as demand increases, employment increases.

$$\frac{\partial L^*}{\partial w} = -2a^2(w + 4b)^{-3} < 0$$

As the cost of labour increases, employment decreases.

3) Consider the consumer problem

$$\min px + qy \quad \text{subject to} \quad x^{0.5} y^{0.5} = U^0$$

where U^0 is a fixed level of utility.

a) Find the consumer's compensated demand functions and check that the second-order conditions for a minimum are satisfied.

b) Derive the expenditure function e , and use the envelope theorem to show that $\partial e / \partial U^0 = \mu^*$.

$$L = px + qy + \lambda(U^0 - x^{0.5}y^{0.5})$$

First-order conditions:

$$L_x = p - 0.5\lambda x^{-0.5}y^{0.5} = 0$$

$$L_y = q - 0.5\lambda x^{0.5}y^{-0.5} = 0$$

$$L_\lambda = U^0 - x^{0.5}y^{0.5} = 0$$

Solve for x^*, y^*, λ^* to obtain:

The compensated (or Hicksian) demand functions:

$$x^* = U^0 \left(\frac{q}{p} \right)^{0.5}$$

$$y^* = U^0 \left(\frac{p}{q} \right)^{0.5}$$

The Lagrange multiplier:

$$\lambda^* = 2(pq)^{0.5}$$

Second-order conditions:

$$L_{xx} = 0.25\lambda x^{-1.5}y^{0.5}$$

$$L_{yy} = 0.25\lambda x^{0.5}y^{-1.5}$$

$$L_{xy} = L_{yx} = -0.25\lambda x^{-0.5}y^{-0.5}$$

$$g_x = 0.5x^{-0.5}y^{0.5}$$

$$g_y = 0.5x^{0.5}y^{-0.5}$$

The Bordered Hessian of the Lagrangian function:

$$H^B = \begin{bmatrix} 0 & 0.5x^{-0.5}y^{0.5} & 0.5x^{0.5}y^{-0.5} \\ 0.5x^{-0.5}y^{0.5} & 0.25\lambda x^{-1.5}y^{0.5} & -0.25\lambda x^{-0.5}y^{-0.5} \\ 0.5x^{0.5}y^{-0.5} & -0.25\lambda x^{-0.5}y^{-0.5} & 0.25\lambda x^{0.5}y^{-1.5} \end{bmatrix}$$

Check the signs of the leading principal minors:

$$|H_1^B| = -0.25x^{-1}y < 0 \quad \text{since } x^*, y^* > 0$$

$$|H_2^B| = -0.125\lambda x^{0.5}y^{-0.5} < 0 \quad \text{since } x^*, y^*, \lambda^* > 0$$

Therefore we have a local minimum

Construct the expenditure function e :

$$e = px^* + qy^*$$

$$\begin{aligned} e &= pU^0 \left(\frac{q}{p}\right)^{0.5} + qU^0 \left(\frac{p}{q}\right)^{0.5} \\ &= 2U^0(pq)^{0.5} \end{aligned}$$

Therefore:

$$\frac{\partial e}{\partial U^0} = 2(pq)^{0.5} = \lambda^*$$

Shepard's Lemma

$$\frac{\partial e}{\partial p} = x^*$$

$$\frac{\partial e}{\partial q} = y^*$$

$$\frac{\partial e}{\partial p} = U^0 p^{-0.5} q^{0.5} = x^*$$

$$\frac{\partial e}{\partial q} = U^0 p^{0.5} q^{-0.5} = y^*$$