EC351: Answers to Problem Set 4

1)

$L = (x_1 + a)x_2^{\,b} + \lambda[m - p_1x_1 - p_2x_2]$

K-T conditions:

$\frac{\partial L}{\partial x_1} = (x_2)^b - \lambda p_1 \le 0 \qquad x_1 \ge 0 \qquad x_1[(x_2)^b - \lambda p_1] = 0$

$\frac{\partial L}{\partial x_2} = b(x_1 + a)(x_2)^{b-1} - \lambda p_2 \le 0 \qquad x_2 \ge 0 \qquad x_2[b(x_1 + a)(x_2)^{b-1} - \lambda p_2] = 0$

$\frac{\partial L}{\partial \lambda} = m - p_1x_1 - p_2x_2 \ge 0 \qquad \lambda \ge 0 \qquad \lambda[m - p_1x_1 - p_2x_2] = 0$

Differently from optimization problems with no constrains or with equality constraints, in a concave programming setting there is not a unique strategy to tackle the problem. The basic idea is to guess and try among all the possibilities that will arise from the non-negativity constraints.
One possible starting point is to set the Lagrange multiplier(s) equal to zero and check what this implies for the K-T conditions.

In our case, if $\lambda = 0$, then from the first condition, we have that $(x_2)^b \le 0$, but by the non-negativity constraint it implies $x_2^{*} = 0$.
When $x_2^{*} = 0$, our utility function is ZERO.
However, a zero utility value cannot be a solution for our problem if income is positive. Indeed, if income is positive and since the utility function is increasing in consumption, we can always increase the value of utility by consuming some positive amount of x_2 and still satisfying the budget constraint.
Therefore, $x_2^{*} = 0$ cannot be a solution and this implies that our guess about the Lagrange multiplier was wrong and therefore λ must be strictly greater than zero.

When $\lambda > 0$, by the last complementary slackness condition, we have that the constraint is binding at the solution, so we can write it as an equality:

$$m - p_1x_1 - p_2x_2 = 0$$

Now check the following cases implied by $x_1 \geq 0$ and $x_2 \geq 0$

(i) $x_1 > 0, x_2 > 0$

This guess assumes that at the solution both goods are consumed in strictly positive quantities.

In this case, we have the usual problem, since we are looking for an interior solution with an equality constraint. The first order conditions are:

$$(x_2)^b - \lambda p_1 = 0$$
$$b(x_1 + a)(x_2)^{b-1} - \lambda p_2 = 0$$
$$m - p_1 x_1 - p_2 x_2 = 0$$

Solving that system of 3 equations in 3 variables we get:

$$x_1^* = \frac{m - abp_1}{(1+b)\,p_1}$$

$$x_2^* = \frac{(m + ap_1)b}{(1+b)\,p_2}$$

$$\lambda^* = \frac{(x_2^*)^b}{p_1}$$

We can check that this solution satisfies all the K-T conditions only if

$m > abp_1$, otherwise $x^*_1 \leq 0$.

Therefore this represents a solution to our maximisation problem.

We also illustrate cases that we already know cannot be a solution just to see how in such cases at least one of the K-T conditions does not hold.

(ii) $x_1 > 0, x_2 = 0$

We know already that this case cannot occur in equilibrium, even if the constraint is binding. However, we can see this more formally by looking at the K-T conditions and finding at least one that is not satisfied.

In particular, if $x_1 > 0$, from the first complementary slackness condition it must be true that:

$$(x_2)^b - \lambda p_1 = 0$$

Since $x_2 = 0$, then $-\lambda p_1 = 0$ that is going to be true only the price of good 1 is zero, since $\lambda > 0$.

However, if $p_1 = 0$ and $x_2 = 0$, from the last complementary slackness condition it must be true that: $m = 0$, that is clearly not possible since income must be positive.

(iii) $x_1 = 0, x_2 = 0$

Since the constraint is binding, this case cannot arise since it violates the budget constraint.

(iv) $x_1 = 0, x_2 > 0$
From the K-T conditions, this case implies that:

$$\begin{aligned} &(x_2)^b - \lambda p_1 < 0 \\ &ba(x_2)^{b-1} - \lambda p_2 = 0 \\ &m - p_2 x_2 = 0 \end{aligned}$$

From the last equation we have that:

$$x_2^* = \frac{m}{p_2} > 0$$

Using that into the second condition we have:

$$\lambda^* = \frac{ab(x_2^*)^{b-1}}{p_2} > 0$$

Let's check if also the first condition is also true:

$$\begin{aligned} &(x_2^*)^b - \frac{ab(x_2^*)^{b-1}}{p_2} p_1 < 0 \\ &\Rightarrow 1 - \frac{ab(x_2^*)^{-1}}{p_2} p_1 < 0 \\ &\Rightarrow 1 < \frac{ab(x_2^*)^{-1}}{p_2} p_1 \end{aligned}$$

$$\begin{aligned} &\Rightarrow x_2^* < \frac{ab}{p_2} p_1 \\ &\Rightarrow \frac{m}{p_2} < \frac{ab}{p_2} p_1 \end{aligned}$$

$$m < abp_1$$

Therefore the K-T conditions are satisfied if $m < abp_1$.

Economic Interpretation:
The utility function implies that the consumer will always consume x_2, otherwise utility would be zero, which cannot be an optimal solution if income is positive. Whether the consumer purchases x_1 depends on its price and on the level of income:

For a given level of income:

If p_1 is relatively low such that $m > abp_1$ then the consumer will consume also x_1 and we obtain an interior solution (case i)).

If p_1 is relatively high such that $m \leq abp_1$ then all the income is spent for x_2 and we have a corner solution (case iv)).
Another interpretation is based on the level of income for a given level of p_1.
If income is sufficiently large, then the consumer consumes both goods.

If the level of income is relatively low, the consumer consumes only good x_2.

2)
(a) The Lagrangean function is

$$L = 40L_1 - 2L_1^2 + 40L_2 - 3.75L_2^2 + \lambda[12 - L_1 - L_2]$$

Kuhn-Tucker conditions:

$$\frac{\partial L}{\partial L_1} = 40 - 4L_1 - \lambda \leq 0 \qquad L_1 \geq 0 \qquad L_1[40 - 4L_1 - \lambda] = 0$$

$$\frac{\partial L}{\partial L_2} = 40 - 7.5L_2 - \lambda \leq 0 \qquad L_2 \geq 0 \qquad L_2[40 - 7.5L_2 - \lambda] = 0$$

$$\frac{\partial L}{\partial \lambda} = 12 - L_1 - L_2 \geq 0 \qquad \lambda \geq 0 \qquad \lambda[12 - L_1 - L_2] = 0$$

(i) First guess that $\lambda = 0$, so that the constraint is not binding at the solution.

In this case:

$$40 - 4L_1 \leq 0$$
$$40 - 7.5L_2 \leq 0$$
$$12 - L_1 - L_2 > 0$$

From the first two inequalities above we have that:

$L_1 \geq 10$, $L_2 \geq 5.3$

Therefore we should have $L_1 + L_2 \geq 15.3$. This result violates the constraint that is

$L_1 + L_2 < 12$

Therefore, our guess $\lambda = 0$ is not correct. The constraint must be binding at the solution and therefore $\lambda > 0$

(ii) $\lambda > 0$, $L_1 = 0, L_2 = 0$

In this case, the constraint will be violated.

Thus, this case cannot be a solution.

iii) $\lambda > 0$ $L_1 > 0, L_2 = 0$

In this case, from the constraint you can find $L_1^* = 12$

Since $L^*_1 = 12$ is positive, the complementary slackness condition $L_1[40 - 4L_1 - \lambda] = 0$ implies that $40 - 4(12) - \lambda = 0$

However, the only way to satisfy that condition is that $\lambda^* = -8$ which is not possible since we know that $\lambda > 0$.

iv) $\lambda > 0$ $L_1 = 0, L_2 > 0$

Using the same reasoning as in iii) you can find that also this case does not satisfy all the K-T conditions simultaneously.

v) $\lambda > 0$ $L_1 > 0, L_2 > 0$

In this case, we are looking at the interior solution where the constraint is binding at that solution:

$40 - 4L_1 - \lambda = 0$

$40 - 7.5L_2 - \lambda = 0$

$12 - L_1 - L_2 = 0$

Get rid of the multiplier from the first two equations:

$40 - 4L_1 = \lambda$

$40 - 7.5L_2 = \lambda$

Therefore: $40 - 4L_1 = 40 - 7.5L_2$

Using this equation together with the constraint you have 2 linear equations in two variables. Solving for L_1 and L_2 we get: $L^*_1 = 7.83$ and $L^*_2 = 4.17$

The Lagrange Multiplier is: $\lambda^* = 8.7$

You can easily check that the K-T conditions are all satisfied at that point.

b) The problem is the same as before apart the total labour supply that now is 20.

Lagrange function:

$$L = 40L_1 - 2L_1^2 + 40L_2 - 3.75L_2^2 + \lambda[20 - L_1 - L_2]$$

K-T conditions:

$$\frac{\partial L}{\partial L_1} = 40 - 4L_1 - \lambda \le 0 \qquad L_1 \ge 0 \qquad L_1[40 - 4L_1 - \lambda] = 0$$

$$\frac{\partial L}{\partial L_2} = 40 - 7.5L_2 - \lambda \le 0 \qquad L_2 \ge 0 \qquad L_2[40 - 7.5L_2 - \lambda] = 0$$

$$\frac{\partial L}{\partial \lambda} = 20 - L_1 - L_2 \ge 0 \qquad \lambda \ge 0 \qquad \lambda[20 - L_1 - L_2] = 0$$

(i) Let's guess first that $L_1 > 0, L_2 > 0$ and $\lambda > 0$ as we found before:

If L_1, L_2 and λ are strictly positive, we are looking for an interior solution and then, the partial derivatives of the Lagrange function must be equal to zero:

$$40 - 4L_1 - \lambda = 0$$

$$40 - 7.5L_2 - \lambda = 0$$

$$20 - L_1 - L_2 = 0$$

The solution of that system is give by: $L^*_1 = 13.04, L^*_2 = 6.96$ and $\lambda^* = -12.2$.

Thus, our result violates the non-negativity constraint on λ. Therefore, we must have that $\lambda = 0$, meaning that the constraint is not binding.

(ii) Let's guess $L_1 > 0, L_2 > 0$ and $\lambda = 0$

$$40 - 4L_1 = 0$$
$$40 - 7.5L_2 = 0$$
$$20 - L_1 - L_2 > 0$$

Then we have : $L^*_1 = 10, L^*_2 = 5.33\overline{3}$

<u>Economic interpretation:</u>

In problem a), the constraint binds which implies that all available labour is used in the production process. In problem b) the amount of available labour is larger than the maximum amount needed for the production (15.3333). Thus, the constraint does not bind and part of the available labour is unemployed.

3)

The Lagrangean

$L = x_1 x_2 + \lambda_1[100 - 2x_1 - 3x_2] + \lambda_2[80 - x_1 - 4x_2]$:

K-T conditions:

$\frac{\partial L}{\partial x_1} = x_2 - 2\lambda_1 - \lambda_2 \leq 0 \qquad x_1 \geq 0 \qquad x_1[x_2 - 2\lambda_1 - \lambda_2] = 0$

$\frac{\partial L}{\partial x_1} = x_1 - 3\lambda_1 - 4\lambda_2 \leq 0 \qquad x_2 \geq 0 \qquad x_2[x_1 - 3\lambda_1 - 4\lambda_2] = 0$

$\frac{\partial L}{\partial \lambda_1} = 100 - 2x_1 - 3x_2 \geq 0 \qquad \lambda_1 \geq 0 \qquad \lambda_1[100 - 2x_1 - 3x_2] = 0$

$\frac{\partial L}{\partial \lambda_2} = 80 - x_1 - 4x_2 \geq 0 \qquad \lambda_2 \geq 0 \qquad \lambda_2[80 - x_1 - 4x_2] = 0$

We assume that neither of the x-values are zero or otherwise utility would be zero.

(i) First guess: $x_1 > 0, x_2 > 0, \lambda_1 > 0$ and $\lambda_2 = 0$:

$$x_2 - 2\lambda_1 = 0$$

$$x_1 - 3\lambda_1 = 0$$

$$100 - 2x_1 - 3x_2 = 0$$

$$80 - x_1 - 4x_2 > 0$$

Solving we obtain: $x^*_1 = 25, x^*_2 = 16.667, \lambda^*_1 = 8.333$

However, the last condition is violated at that solution since we have:

$80 - 25 - 4(16.667) = -11.667 < 0$

Our first guess is not the right one.

(ii) Now guess $x_1 > 0, x_2 > 0, \lambda_1 = 0$ and $\lambda_2 > 0$

This implies:

$$x_2 - \lambda_2 = 0$$

$$x_1 - 4\lambda_2 = 0$$

$$100 - 2x_1 - 3x_2 > 0$$

$$80 - x_1 - 4x_2 = 0$$

Solving we obtain:

$x^*_1 = 40, x^*_2 = 10, \lambda^*_2 = 10$

But again, looking at the budget constraint we have:

$100 - 80 - 30 = -10 < 0$

Again, our guess is not the right one.

(iii) Finally, assume that $x_1 > 0, x_2 > 0, \lambda_1 > 0$ and $\lambda_2 > 0$.

This implies:

$$x_2 - 2\lambda_1 - \lambda_2 = 0$$

$$x_1 - 3\lambda_1 - 4\lambda_2 = 0$$

$$100 - 2x_1 - 3x_2 = 0$$

$$80 - x_1 - 4x_2 = 0$$

Solving that system we obtain (for example, use the last two constraints to find the values of x_1 and x_2 and then use the first two equations above to find the multipliers):

$x^*_1 = 32, x^*_2 = 12, \lambda^*_1 = 3.2, \lambda^*_2 = 5.6$

We can check that this is the right solution since it satisfies the K-T conditions.