

EC351: Answers to Problem Set 5

1)

We have the following:

Supply: $Q_t = F + Gp_t$

Demand: $Q_t = A + Bp_t + \theta Q_{t-1}$

The inverse supply is: $p_t = \frac{Q_t - F}{G}$

Use that to eliminate p_t from the demand equation:

$$Q_t = A + B\left(\frac{Q_t - F}{G}\right) + \theta Q_{t-1}$$

$$Q_t = A + \frac{B}{G}Q_t - \frac{BF}{G} + \theta Q_{t-1}$$

$$Q_t - \frac{B}{G}Q_t = A - \frac{BF}{G} + \theta Q_{t-1}$$

$$Q_t\left(\frac{G - B}{G}\right) = \frac{AG - BF + G\theta Q_{t-1}}{G}$$

The first order difference equation for the quantity is therefore:

$$Q_t = \frac{AG - BF}{G - B} + \frac{G\theta}{G - B}Q_{t-1} \quad 1)$$

To solve equation 1) we use the following steps:

- We find the **PARTICULAR SOLUTION** (or **Steady-State** solution) of equation 1);
- We find the **COMPLEMENTARY SOLUTION**, that is the solution of the **HOMOGENEOUS** equation associated with equation 1);
- The **GENERAL SOLUTION** is simply the sum of the complementary and particular solution.
- We use the initial condition (if we have it) to find the unique value of the arbitrary constant that appears in the General Solution.

Particular Solution:

The particular solution is found by setting: $Q_t = Q_{t-1} = \bar{Q}$ (a constant). This implies that we are looking for the solution of equation 1) in which the variable Q does not change over time. The particular solution will be the steady-state (or **long run equilibrium**) of our variable Q according to its law of motion (given by the first order difference equation).

$$\bar{Q} = \frac{GA - BF}{G - B} + \frac{\theta G}{G - B} \bar{Q}$$

$$\Rightarrow \bar{Q} = \frac{GA - BF}{G(1 - \theta) - B}$$

Notice that the steady-state solution is positive since B is assumed to be negative.

Complementary Solution:

The complementary solution is the solution of the HOMOGENEOUS equation associated with our first-order difference equation (= equation 1) without any constant term that is not attached to the variable Q). In our case the homogeneous equation is simply:

$$Q_t = \frac{\theta G}{G - B} Q_{t-1}$$

The solution of the equation above is given by:

$$Q_t = \left(\frac{\theta G}{G - B} \right)^t C$$

Where C is an arbitrary constant.

Thus, the general solution of our first-order difference equation is:

$$GS = CS + PS$$

$$Q_t = \left(\frac{\theta G}{G - B} \right)^t C + \frac{GA - BF}{G(1 - \theta) - B}$$

If we know the initial condition Q_0 we can find the unique value of the constant C for our equation.

To do that: evaluate the general solution at time $t=0$:

$$Q_0 = \left(\frac{\theta G}{G - B} \right)^0 C + \frac{GA - BF}{G(1 - \theta) - B}$$

Therefore the value of the constant C is given by:

$$C = Q_0 - \frac{GA - BF}{G(1 - \theta) - B}$$

In a first-order linear difference equation the constant is always equal to the difference between the initial condition and the steady-state solution. Indeed, the constant represents the distance between the initial point (the initial condition) and the point where the equation should tend to (the steady-state).

Therefore, our solution can be written as:

$$Q_t = \left(\frac{\theta G}{G - B} \right)^t \left(Q_0 - \frac{GA - BF}{G(1 - \theta) - B} \right) + \frac{GA - BF}{G(1 - \theta) - B}$$

Does Q_t as defined by the solution above converge to the steady-state?

Convergence requires that:

$$\left| \frac{\theta G}{G - B} \right| < 1$$

because only in that case, as $t \rightarrow \infty$ the term $\left(\frac{\theta G}{G - B} \right)^t$ in the solution above will tend to zero and so the solution for Q_t will converge to the steady-state $\frac{GA - BF}{G(1 - \theta) - B}$.

Given that $G > 0, B < 0$ and $0 < \theta < 1$, then it must be true that $0 < \frac{\theta G}{G - B} < 1$, and therefore Q will converge MONOTONICALLY to the steady-state.

2)

Demand=Supply implies:

$$\begin{aligned} \theta - \sigma p_{t+1} &= \frac{N(p_t - \alpha)}{2\beta} \\ \Rightarrow p_{t+1} &= \frac{2\beta\theta + N\alpha}{2\beta\sigma} - \frac{N}{2\beta\sigma} p_t \end{aligned}$$

As before: the particular solution is given by the solution of the difference equation when $p_{t+1} = p_t = \bar{p}$:

$$\begin{aligned} \bar{p} &= \frac{2\beta\theta + N\alpha}{2\beta\sigma} - \frac{N}{2\beta\sigma} \bar{p} \\ \Rightarrow \bar{p} &= \frac{2\beta\theta + N\alpha}{2\beta\sigma + N} \end{aligned}$$

The complementary solution is the solution of the homogeneous equation associated with the difference equation:

$$p_{t+1} = -\frac{N}{2\beta\sigma} p_t$$

That is:

$$p_t = \left(-\frac{N}{2\beta\sigma}\right)^t C$$

Therefore, the general solution is given by:

$$p_t = \left(-\frac{N}{2\beta\sigma}\right)^t C + \frac{2\beta\theta + N\alpha}{2\beta\sigma + N}$$

If we know the initial condition p_0 we can find the value of C: for a first-order difference equation is always: p_0 minus the particular solution.

Therefore, the general solution of our difference equation is:

$$p_t = \left(-\frac{N}{2\beta\sigma}\right)^t \left(p_0 - \frac{2\beta\theta + N\alpha}{2\beta\sigma + N}\right) + \frac{2\beta\theta + N\alpha}{2\beta\sigma + N}$$

Stability: does p_t converges to its steady state?

Again, we need to check if:

$$\left|-\frac{N}{2\beta\sigma}\right| < 1$$

Since only in that case: $\lim_{t \rightarrow \infty} \left(-\frac{N}{2\beta\sigma}\right)^t = 0$

That ratio is less than one in absolute value if:

$$N < 2\beta\sigma$$

In that case:

$$-1 < -\frac{N}{2\beta\sigma} < 0$$

Therefore, p_t will converge to its steady-state in an oscillatory manner (non-monotonic way) when $N < 2\beta\sigma$.

3)

Use the first two equations to eliminate Q and S from the third one:

$$p_{t+1} = a(c - g) + [1 - a(h - z)]p_t$$

The particular solution is:

$$\bar{p} = \frac{c - g}{h - z}$$

The complementary solution is the solution of:

$$p_{t+1} = [1 - a(h - z)]p_t$$

That is:

$$p_t = [1 - a(h - z)]^t C$$

Therefore, the general solution is:

$$p_t = [1 - a(h - z)]^t C + \frac{c - g}{h - z}$$

If we know the initial condition p_0 , we can find the value of K:

$$p_t = [1 - a(h - z)]^t \left(p_0 - \frac{c - g}{h - z} \right) + \frac{c - g}{h - z}$$

Convergence requires: $|1 - a(h - z)| < 1$

Since $a, h > 0$ and $z < 0$:

If: $0 < a(h - z) < 1$

there is a monotonic convergence to the steady-state.

If $1 < a(h - z) < 2$

Then, p converges with oscillations to the steady-state.

4)

To solve the problem, we need to use the classical equilibrium condition: $Y=C+I$

Therefore:

$$Y_t = C_t + I_t$$

$$Y_t = 200 + 0.75Y_{t-1} + 50 + 0.15Y_{t-1}$$

$$\Rightarrow Y_t = 250 + 0.9Y_{t-1}$$

The particular solution is when: $Y_t = Y_{t-1} = \bar{Y}$ in the previous difference equation:

$$\bar{Y} = 250 + 0.9\bar{Y}$$

$$\Rightarrow \bar{Y} = \frac{250}{0.1} = 2500$$

The complementary solution is the solution of:

$$Y_t = 0.9Y_{t-1}$$

$$\Rightarrow Y_t = (0.9)^t K$$

Therefore, the general solution is:

$$Y_t = (0.9)^t K + 2500$$

Since we know the initial condition $Y_0=3000$, we can find the value of K :

At time $t=0$:

$$Y_0 = (0.9)^0 K + 2500$$

$$\Rightarrow K = Y_0 - 2500$$

$$\Rightarrow K = 500$$

Therefore, our general solution is:

$$Y_t = (0.9)^t 500 + 2500$$

Since 0.9 is less than 1 (and greater than zero), Y_t will converge monotonically to the steady-state level given by 2500.