EC351: Answers to Problem Set 6

1)

a) $$y_{t+1} = \frac{3}{16} + y^2{}_t$$

This is a non-linear first order difference equation.

We can find the steady-states of that equation and see the local stability properties of those equilibria.

The steady-state is defines as: $y_{t+1} = y_t = \overline{y}$

Using this fact into the original equation:

$$\overline{y} = \frac{3}{16} + \overline{y}^2$$

$$\Rightarrow \overline{y}^2 - \overline{y} + \frac{3}{16} = 0$$

We can solve the quadratic equation above:

$$\overline{y} = \frac{1 \pm \sqrt{1 - 3/4}}{2}$$

$$\Rightarrow \overline{y}_1 = 0.75, \overline{y}_2 = 0.25$$

We have two steady-states.

To see the stability properties of those equilibria we need to evaluate the first derivative of the original function at each equilibrium value.

$$f'(y_t) = 2y_t$$

The first derivative is a straight line positively sloped.

When $\overline{y} = 0.75$:

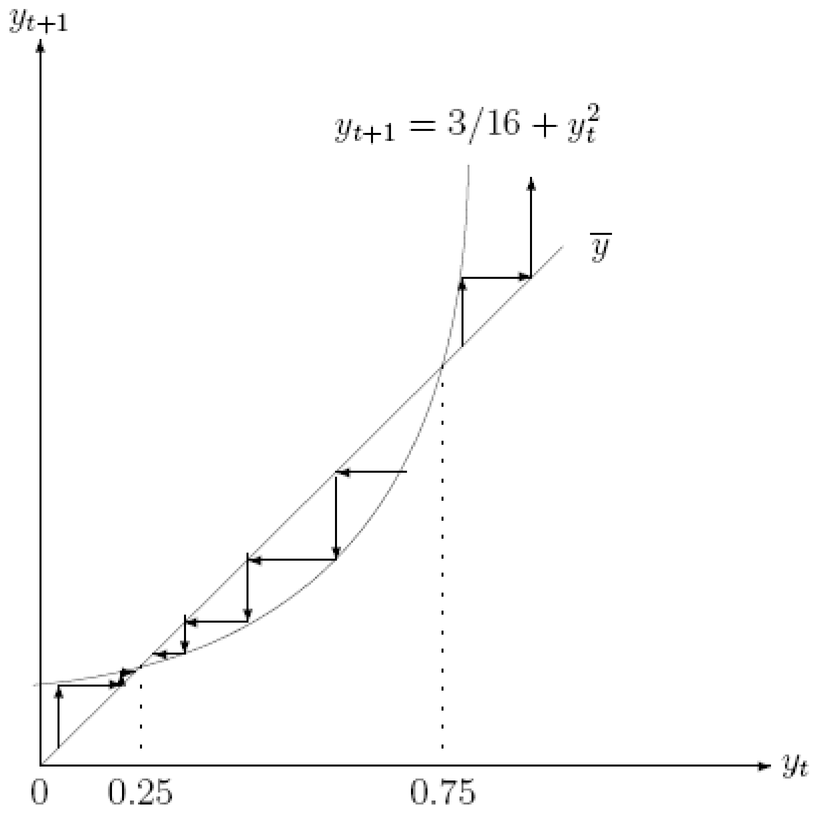
$$f'(\overline{y}) = 2(0.75) = 1.5$$

Since 1.5 > 1, then the steady-state $\overline{y} = 0.75$ is locally unstable.

When $\overline{y} = 0.25$: $f'(\overline{y}) = 2(0.25) = 0.5$

Since 0.5 < 1 the steady-state $\overline{y} = 0.25$ is locally stable.

The phase diagram:



b)

$$y_{t+1} = 4 + \frac{9}{4y_t}$$

Again, we have a non-linear first order difference equation.

Solve for the steady-state: $y_{t+1} = y_t = \overline{y}$

$$\overline{y} = 4 + \frac{9}{4\overline{y}}$$

$$\Rightarrow \overline{y}^2 - 4\overline{y} - \frac{9}{4} = 0$$

Solve the quadratic equation above:

$$\bar{y} = \frac{4 \pm \sqrt{16+9}}{2}$$
$$\Rightarrow \bar{y}_1 = 4.5, \bar{y}_2 = -0.5$$

The first order derivative is:

$$f^{'}(y_t) = -\frac{9}{4} y_t^{-2}$$

Evaluate the first derivative at $\bar{y} = 4.5$

$$f^{'}(\bar{y}) = -\frac{9}{4}(4.5)^{-2} = -\frac{1}{9}$$

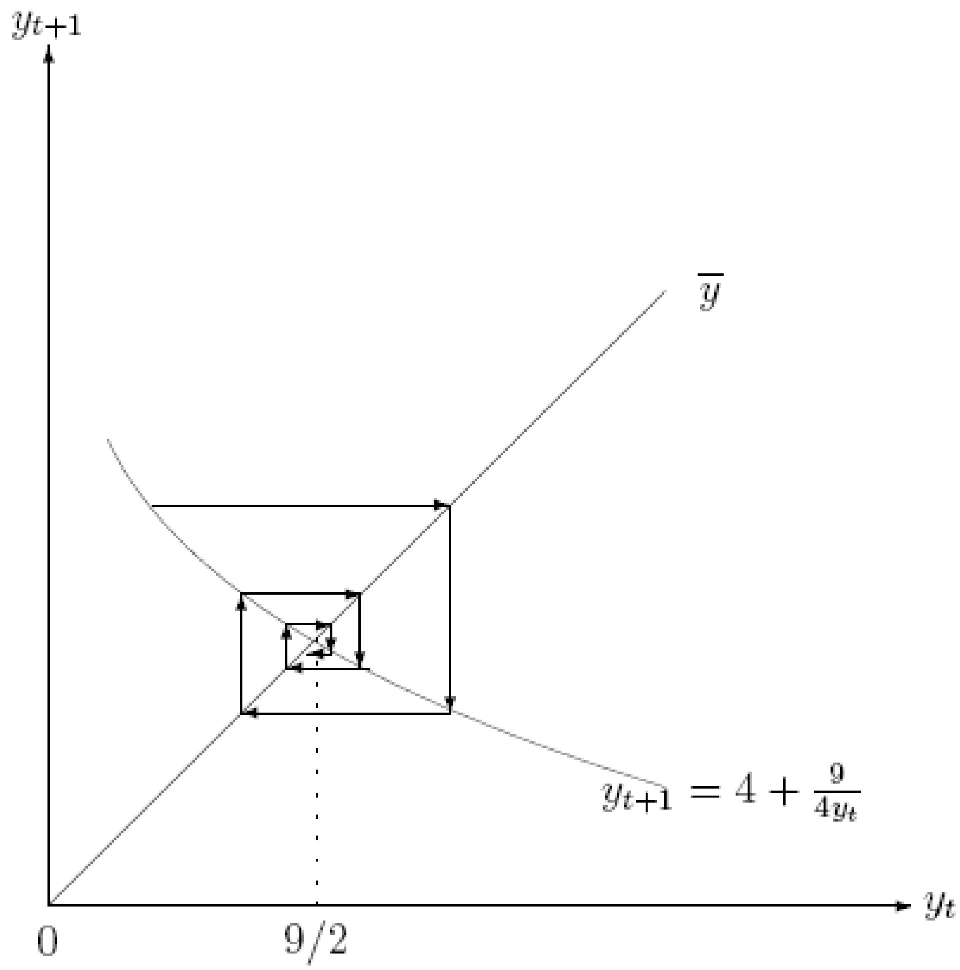
Since $\left|\frac{1}{9}\right| < 1$, the steady-state $\bar{y} = 4.5$ is locally stable.

When $\bar{y} = -0.5$

$$f^{'}(\bar{y}) = -\frac{9}{4}(-0.5)^{-2} = -9$$

At this steady-state the first derivative is equal to -9. This, means that the steady-state $\bar{y} = -0.5$ is not locally stable

Consider the phase diagram in the non-negative orthant only:



2)

The difference equation is given by:

$$k_{t+1} = k_t - \delta k_t + sy_t$$

We know that $sy_t = y_t^{1/2}$. Using the definition of $y_t = k_t^{\alpha}$, we have that:

$$k_{t+1} = k_t - \delta k_t + k_t^{\frac{\alpha}{2}}$$

Find the steady-states: $k_{t+1} = k_t = k$

$$k - k(1-\delta) - k^{\frac{\alpha}{2}} = 0$$

$$k\left[1 - (1-\delta) - k^{\frac{\alpha}{2}-1}\right] = 0$$

$$k\left[1 - (1-\delta) - k^{\frac{\alpha-2}{2}}\right] = 0$$

There are two steady-states:

$k = 0$

$k = \delta^{\frac{2}{\alpha-2}}$

The derivative: $\dfrac{dk_{t+1}}{dk_t} = 1 - \delta + \dfrac{\alpha}{2} k_t^{\frac{\alpha}{2}-1}$

When $k = 0$: $\quad 1 - \delta + \dfrac{\alpha}{2}(0)^{\frac{\alpha}{2}-1}$

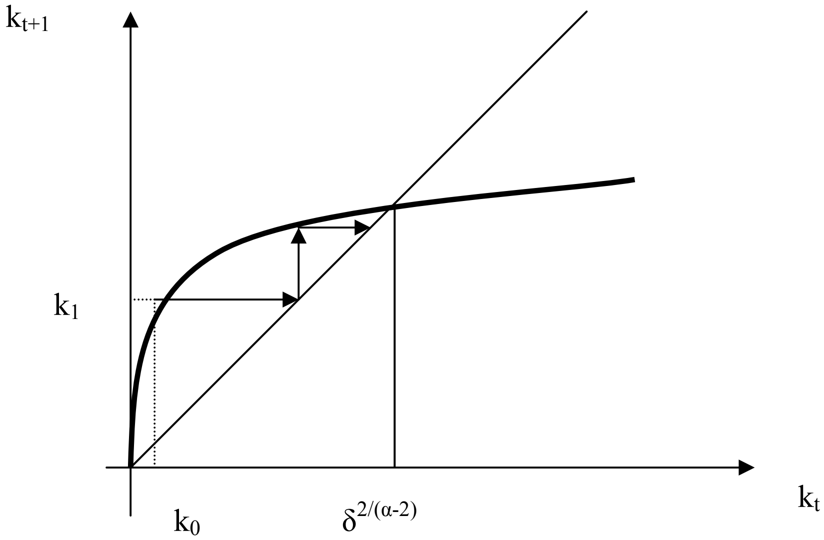
However, since $0 < \alpha < 1$, the exponent $\dfrac{\alpha}{2} - 1$ is negative. Therefore that derivative will tend to infinity as $k \to 0$, so $k = 0$ is not locally stable.

When $k = \delta^{\frac{2}{\alpha-2}}$:

$$1 - \delta + \frac{\alpha}{2}\left(\delta^{\frac{2}{\alpha-2}}\right)^{\frac{\alpha-2}{2}} = 1 - \delta\left(1 - \frac{\alpha}{2}\right)$$

Notice that the term in the brackets is less than one and it is multiplied by δ that is less than one. Therefore $0 < \delta\left(1 - \dfrac{\alpha}{2}\right) < 1$, so the derivative at that steady-state is positive and less than 1. So this steady state is locally stable.

From the phase diagram we can say that $k = \delta^{\frac{2}{\alpha-2}}$ is also globally stable.



3)

a) $$y_{t+2} - y_t = 0$$

This is a second order, linear, homogeneous difference equation.

The **Particular Solution** is zero (just substitute the fact that $y_{t+2} = y_t = \bar{y}$ into the equation above or just notice that the equation above is already homogenous).

To find the **Complementary Solution:**

Write down the characteristic equation of equation a) (since that equation is already homogenous):

$$r^2 - 1 = 0$$

Therefore: $r = \pm 1$.

We have two distinct real roots.

The complementary solution is given by:

$$y_h = C_1 r_1^t + C_2 r_2^t = C_1 (1)^t + C_2 (-1)^t$$

That is also the **General Solution** of our equation.

Note: our solution will not converge to the steady-state (given by $\bar{y} = 0$) as t increases.

We can see this also by looking at the coefficients of the equation.

Convergence requires that:

$$1 + a_1 + a_2 > 0$$
$$1 - a_1 + a_2 > 0$$
$$a_2 < 1$$

In our case we have: $a_1 = 0, a_2 = -1$.

The first two conditions above are not met.

To find C_1 and C_2 we need to know y_0 and y_1 .

b) $$y_{t+2} + 2y_{t+1} + y_t = 16$$

This is a second order, linear, **non-homogeneous** difference equation.

The **Particular Solution** is given by the solution of:

$$\bar{y} + 2\bar{y} + \bar{y} = 16$$
$$\Rightarrow \bar{y} = 4$$

To find the **Complementary Solution**:

Write down the characteristic equation associated with the homogenous version of equation b) (that is: $y_{t+2} + 2y_{t+1} + y_t = 0$):

$$r^2 + 2r + 1 = 0$$

Solving that quadratic equation:

$$r = \frac{-2 \pm \sqrt{4-4}}{2} = -1$$

We have two REPEATED roots.

Therefore the complementary solution is:

$$y_h = C_1(-1)^t + C_2 t(-1)^t$$

The General Solution is then (CS + PS):

$$y_t = C_1(-1)^t + C_2 t(-1)^t + 4$$

The solution does not converge because the real root is on the unit circle (its absolute value is exactly equal to 1. For convergence it should have been strictly lower than 1 in absolute value).

4)

Substitute out C_t and I_t into the national accounting identity:

$$Y_t = mY_t + a(Y_{t-1} - Y_{t-2}) + G$$
$$\Rightarrow Y_t = \frac{a(Y_{t-1} - Y_{t-2}) + G}{1-m}$$

Or, rewriting the above equation (it has to howl for any t):

$$Y_{t+2} = \frac{a}{1-m}Y_{t+1} - \frac{a}{1-m}Y_t + \frac{1}{1-m}G$$
$$\Rightarrow Y_{t+2} - \frac{a}{1-m}Y_{t+1} + \frac{a}{1-m}Y_t = \frac{1}{1-m}G$$

The **particular solution** of that second order difference equation is:

$$\overline{Y} - \frac{a}{1-m}\overline{Y} + \frac{a}{1-m}\overline{Y} = \frac{1}{1-m}G$$
$$\Rightarrow \overline{Y} = \frac{1}{1-m}G$$

To find the **complementary solution**:

The characteristic equation is:

$$r^2 - \frac{a}{1-m}r + \frac{a}{1-m} = 0$$

The solutions are:

$$r = \frac{\frac{a}{(1-m)} \pm \sqrt{\left(\frac{a}{(1-m)}\right)^2 - 4\left(\frac{a}{1-m}\right)}}{2}$$
$$\Rightarrow r = \frac{a}{2(1-m)} \pm \frac{1}{2}\sqrt{\left(\frac{a}{(1-m)}\right)\left[\left(\frac{a}{(1-m)}\right) - 4\right]}$$

It is difficult to see the stability properties from the roots above.

However, we know the stability conditions expressed in terms of the parameters of the original equation:

$$1 + a_1 + a_2 > 0$$
$$1 - a_1 + a_2 > 0$$
$$a_2 < 1$$

In our case we have: $a_1 = -\frac{a}{1-m}$ and $a_2 = \frac{a}{1-m}$

The first condition is satisfied. Since it becomes:

$$1 - \frac{a}{1-m} + \frac{a}{1-m} = 1 > 0$$

The second condition is satisfied as well if a > 0 as it is natural to assume.

$$1 + \frac{2a}{1-m} > 0$$

Therefore, the crucial assumption that must hold for stability is:

$$\frac{a}{1-m} < 1$$

If that condition holds, it means that the roots of the characteristic equation must be complex-valued.

Therefore, the **complementary solution** that satisfies the stability conditions must be:

$$y_h = \left(\frac{a}{1-m}\right)^{\frac{t}{2}} \left[C_1 \cos\theta t + C_2 \sin\theta t\right]$$

The **General Solution** is given by:

$$y_t = \left(\frac{a}{1-m}\right)^{\frac{t}{2}} \left[C_1 \cos\theta t + C_2 \sin\theta t\right] + \frac{G}{1-m}$$

where: $\cos\theta = \frac{-a_1}{2\sqrt{a_2}} = \frac{\frac{a}{1-m}}{2\left(\frac{a}{1-m}\right)^{0.5}} = \frac{\sqrt{\frac{a}{1-m}}}{2}$

and

$$\sin\theta = \frac{\sqrt{4a_2 - a_1^2}}{2\sqrt{a_2}} = \frac{\sqrt{4\frac{a}{1-m} - \left(-\frac{a}{1-m}\right)^2}}{2\sqrt{\frac{a}{1-m}}}$$