EC351: Answers to Problem Set 7

1)

a) $$\int (x^4 + 2x^3 + 4x + 10)dx$$

We can use the property about the integral of a sum and rewrite the above integral as:

$$\int x^4 dx + 2\int x^3 dx + 4\int xdx + 10\int dx$$

$$\left(\frac{x^5}{5} + C_1\right) + 2\left(\frac{x^4}{4} + C_2\right) + 4\left(\frac{x^2}{2} + C_3\right) + 10\left(x + C_4\right)$$

$$\frac{x^5}{5} + \frac{x^4}{2} + 2x^2 + 10x + C$$

Where $C = C_1 + 2C_2 + 4C_3 + 10C_4$

b) $\int x^{\frac{2}{3}} dx$

Use the power rule:

$$\int x^{\frac{2}{3}} dx = \frac{3}{5} x^{\frac{5}{3}} + C$$

c) $$\int 10e^x dx = 10(e^x + C_1) = 10e^x + C$$

where $C = 10C_1$

d) $$\int \frac{(3x^2 + 2)}{(x^3 + 2x)} dx$$

Note that:

$$\int \frac{f'(x)}{f(x)} dx = \ln(f(x)) + C$$

Therefore:

$$\int \frac{(3x^2 + 2)}{(x^3 + 2x)} dx = \ln(x^3 + 2x) + C$$

2) Consider a linear, first-order differential equation (autonomous):

i) $$\dot{y}(t) + ay = b$$

The general solution of that equation is always:

$$y(t) = Ce^{-at} + \frac{b}{a}$$

Given an initial condition $y(0)$, the general solution can be written as:

ii) $$y(t) = \left(y(0) - \frac{b}{a} \right) e^{-at} + \frac{b}{a}$$

a) $\dot{y}(t) - y = 0$

With respect to equation i) here we have a= - 1 and b=0.

Knowing y(0) = 1 and using ii), the general solution of our equation is:

$$y(t) = (1)e^{t}$$

As t increases, y(t) will NOT converge to the steady-state $\bar{y} = 0$.

b) $\dot{y}(t) + 3y = 12$

The particular solution is: $\bar{y} = \frac{12}{3} = 4$

The complementary solution is: $y_h(t) = Ce^{-3t}$

Knowing y(0) = 10, the general solution is:

$y(t) = (10 - 4)e^{-3t} + 4$
$\Rightarrow 6e^{-3t} + 4$

In this case the solution will converge to the steady-state.

c) $2\dot{y}(t) + 0.5y = 12$

Rewrite the above equation as:

$$\dot{y}(t) + 0.25y = 6$$

The particular solution is:

(when $\dot{y}(t) = 0$)

$$0.25\bar{y} = 6$$
$$\Rightarrow \bar{y} = 24$$

The complementary solution is:

(the solution of the homogeneous equation $\dot{y}(t) + 0.25y = 0$)

$$y_h(t) = Ce^{-0.25t}$$

Given y(0) = 10, the general solution is:

$$y(t) = (10 - 24)e^{-0.25t} + 24$$
$$\Rightarrow -14e^{-0.25t} + 24$$

d) $\dot{y}(t) = 5$

In this case we have that the time derivative of y is a constant.

In this case we cannot use the general solution ii) since here we have a=0.

When a=0, the general solution of the equation $\dot{y}(t) = b$, is given by:

iii) $y(t) = bt + y(0)$

Equation iii) is indeed a solution for $\dot{y}(t) = b$ (from iii) $\dot{y}(t) = b$)

Using iii) and given y(0) = 1, the general solution of $\dot{y}(t) = 5$ is given by:

$$y(t) = 5t + 1$$

e) $\dot{y}(t) - 6y = -6$

The particular solution:

$\bar{y} = 1$

The complementary solution:

$y_h(t) = Ce^{6t}$

Given y(0)=3, the general solution is given by:

$y(t) = 2e^{6t} + 1$

3) We know that $\dot{q} = \alpha(P^D - P^S)$

Substituting the expressions for P^D and P^S into the equation above gives us:

$$\dot{q} = \alpha(a + bq - g - hq)$$

Re-arrange the above expression:

$$\dot{q} - \alpha(b - h)q = \alpha(a - g)$$

The particular solution:

When $\dot{q} = 0$ we have the classical equilibrium $P^D = P^S$.

Therefore: $\bar{q} = \dfrac{g - a}{b - h}$

The complementary solution is the solution of

$$\dot{q} - \alpha(b - h)q = 0$$

Therefore:

$$y_h(t) = Ce^{\alpha(b-h)t}$$

Given the initial condition q^0, the general solution can be written as:

$$y(t) = \left(q^0 + \frac{a-g}{b-h}\right)e^{\alpha(b-h)t} + \frac{g-a}{b-h}$$

For stability: assuming $\alpha > 0$, then stability occurs only if $b - h < 0$.

In the usual framework, the inverse demand is negatively sloped ($b<0$) while the inverse supply is positively sloped ($h>0$). Therefore that condition is always met (if $\alpha > 0$).