EC351 Mathematical Economics 2011/2012

Problem Set 9 (Final Problem Set)

1. For each of the following linear differential equation systems, solve using:

   a) The Substitution Method
   b) The Direct Method

i. $\dot{y}_1 = -2y_1 + 2y_2 + 12$

$\dot{y}_2 = y_1 - 3y_2 - 12$

$y_1(0) = -2,$
$y_2(0) = 5$

ii. $\dot{y}_1 = -y_1 - \frac{9}{4}y_2 + 2$

$\dot{y}_2 = -3y_1 + 2y_2 - 1$

$y_1(0) = 20,$
$y_2(0) = 2$

iii. $\dot{y}_1 = 2y_1 - 2y_2 + 5$

$\dot{y}_2 = 2y_1 + 2y_2 + 1$

$y_1(0) = 2.5,$
$y_2(0) = -1$

Ensure that the solutions satisfy the initial conditions given.

2. Solve the following dynamic optimization problem using optimal control theory

$$\text{Maximise} \int_0^5 (3x - y^2)dt$$

$$\text{subject to} \quad \dot{x} = 5y,$$

$$x(0) = 2.$$

3. Solve the following optimal control problem:

$$\text{Maximise} \int_0^T (y - y^2 - 4x - 3x^2)dt$$

$$\text{subject to} \quad \dot{x} = x + y,$$

$$x(0) = x_0 .$$

4. Suppose a firm's only factor of production is its capital stock, *K*. Its production function is given by the relation, *Q* = *5K*, where *Q* is the quantity produced. Capital stock does not depreciate. The change in the capital stock is therefore equal to only the firm's investment, *I*:

$$\dot{K} = I - \delta K$$

where $\delta \in (0,1)$ is the depreciation rate.

The price of the firm's output is p, and the cost of investment, *I*, is given by the relation, $\frac{1}{2}bI^2$ where b is a constant. The firm also incurs a cost on the repairs and maintenance at the rate of *r* on the existing capital stock each period. Therefore the firm's profit at a point in time:

$$\pi = p5K - rK - \frac{1}{2}bI^2 .$$

The firm wants to choose a path of investment to maximise profit over a time period.

$$\text{Maximise} \int_0^T (5pK - rK - \frac{1}{2}bI^2)dt$$

$$\text{subject to} \quad \dot{K} = I - \delta K,$$

$$K(0) = K_0 .$$

Solve for the optimal paths of the control variable.