TEST AUTUMN 2011
SOLUTION

QUESTION 1: $\min_{x,y} E = P_x \cdot x + P_y \cdot y$ s.t. $x^\alpha \cdot y^{1-\alpha} = \bar{U}$

WITH $\alpha > 0$ BUT LESS THAN 1. $\bar{U} > 0$.

PART a):

LAGRANGEAN: $\mathcal{L} = P_x \cdot x + P_y \cdot y + \lambda\left[\bar{U} - x^\alpha y^{1-\alpha}\right]$

F.O.C.s:

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow P_x - \alpha \lambda x^{\alpha-1} \cdot y^{1-\alpha} = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0 \Rightarrow P_y - (1-\alpha)\lambda x^\alpha y^{-\alpha} = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow x^\alpha \cdot y^{1-\alpha} = \bar{U} \quad (3)$$

PART b): DIVIDE (1) AND (2) (TO GET RID OF λ):

$$\frac{P_x}{P_y} = \frac{\alpha}{1-\alpha} \cdot \frac{x^{\alpha-1}}{x^\alpha} \cdot \frac{y^{1-\alpha}}{y^{-\alpha}}$$

FROM THAT:

$$\frac{P_x}{P_y} = \frac{\alpha}{1-\alpha} \cdot \frac{1}{x^\alpha \cdot x^{-\alpha+1}} \cdot y^{1-\alpha} \cdot y^\alpha$$

$$\Rightarrow \frac{P_x}{P_y} = \frac{\alpha}{1-\alpha} \cdot \frac{y}{x} \Rightarrow y = \frac{P_x}{P_y} \cdot \frac{1-\alpha}{\alpha} \cdot x \quad (4)$$

USE (4) INTO (3) TO SOLVE FOR x:

$$\Rightarrow x^{\alpha}\left(\frac{P_X}{P_Y}\right)^{1-\alpha}\cdot\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}\cdot x^{1-\alpha}=\bar{U}$$

$$\Rightarrow x^{\alpha}\cdot x^{1-\alpha}\cdot\left(\frac{P_X}{P_Y}\right)^{1-\alpha}\cdot\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}=\bar{U}$$

$$\Rightarrow x^{*}=\bar{U}\cdot\left(\frac{P_Y}{P_X}\right)^{1-\alpha}\cdot\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}$$

COMPENSATED DEMAND FOR GOOD x: AN INCREASE IN THE STANDARD OF LIVING $(\bar{U})$ WILL INCREASE THE DEMAND FOR GOOD x. AN INCREASE IN P_X WILL MAKE GOOD x RELATIVELY MORE EXPENSIVE AND IN ORDER TO STAY ON THE SAME INDIFFERENCE CURVE ($\bar{U}$ IS GIVEN) THE DEMAND FOR x SHOULD DECREASE. AN INCREASE IN P_Y MAKES GOOD x RELATIVELY CHEAPER INSTEAD.

ONCE YOU HAVE THE SOLUTION FOR x SUBSTITUTE IT INTO (4) AND SOLVE FOR Y:

$$\Rightarrow Y=\frac{P_X}{P_Y}\cdot\frac{1-\alpha}{\alpha}\cdot\bar{U}\cdot\left(\frac{P_Y}{P_X}\right)^{1-\alpha}\cdot\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}$$

$$\Rightarrow Y=P_X\cdot P_X^{\alpha-1}\cdot P_Y^{-1}\cdot P_Y^{1-\alpha}\cdot(1-\alpha)(1-\alpha)^{\alpha-1}\cdot\alpha^{-1}\cdot\alpha^{1-\alpha}\cdot\bar{U}$$

$$\Rightarrow Y^{*}=\bar{U}\cdot\left(\frac{P_X}{P_Y}\right)^{\alpha}\cdot\left(\frac{1-\alpha}{\alpha}\right)^{\alpha}$$

$\Rightarrow$ SIMILAR INTERPRETATION AS BEFORE.

PART c) : IS THE SOLUTION IN b) LOCAL OR GLOBAL?

USE THEOREM 2 IN LECTURE 2.

THE OBJECTIVE FUNCTION (E) IS LINEAR THEREFORE IT IS CONVEX/CONCAVE AND SO QUASICONVEX/QUASICONCAVE.

IF THE CONSTRAINT IS STRICTLY QUASICONCAVE THE SOLUTION IS UNIQUE AND THEREFORE A GLOBAL MINIMUM.

THE CONSTRAINT IS $U(x,y) - \bar{U} = 0$ WHERE $U(x,y) = x^{\alpha} y^{1-\alpha}$

WE KNOW THAT $\left(\text{SINCE } 0<\alpha<1 \text{ AND } x,y>0\right)$: $U_x,\ U_y > 0$ (FIRST ORDER DERIVATIVES OF U)

NON NEGATIVE ORTHANT

$U_{xx} < 0,\ U_{yy} < 0$ AND $U_{yx} = U_{xy} > 0$

(SECOND ORDER DERIVATIVES)

BORDERED HESSIAN :

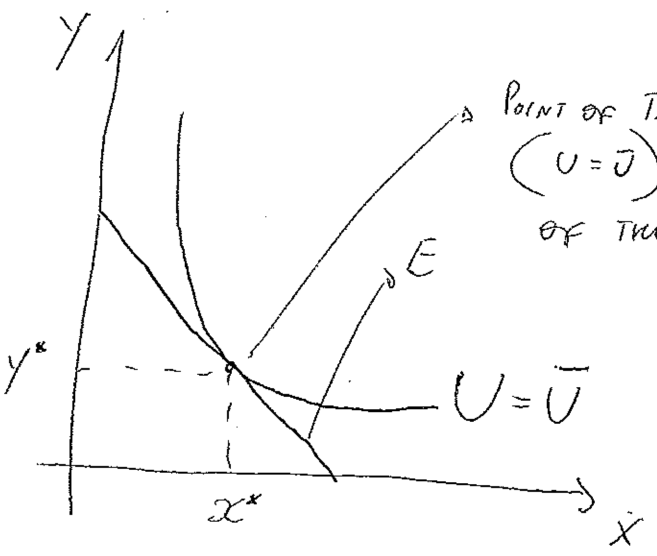
$$\bar{H} = \begin{bmatrix} 0 & U_x & U_y \\ U_x & U_{xx} & U_{xy} \\ U_y & U_{yx} & U_{yy} \end{bmatrix}$$

LEADING PRINCIPAL MINORS : $\bar{H}_1 = \begin{vmatrix} 0 & U_x \\ U_x & U_{xx} \end{vmatrix} = -U_x^2 < 0$

$$|H_2| = -\underbrace{U_x \cdot \begin{vmatrix} U_x & U_y \\ U_{yx} & U_{yy} \end{vmatrix}}_{+} + \underbrace{U_y \cdot \begin{vmatrix} U_x & U_y \\ U_{xx} & U_{xy} \end{vmatrix}}_{+} > 0$$

THE CONSTRAINT IS STRICTLY QUASICONCAVE. THE SOLUTION IN b) IS UNIQUE AND THEREFORE IS A GLOBAL MINIMUM.

GRAPH



POINT OF TANGENCY BETWEEN THE CONSTRAINT $(U=\bar{U})$ AND ~~[illegible]~~ THE LOWEST LEVEL CURVE OF THE OBJECTIVE FUNCTION.

PART d)

DENOTE WITH $E^* = P_x \cdot x^* + P_y \cdot y^*$ THE VALUE FUNCTION OF THIS PROBLEM.

SHEPHARD'S LEMMA: $\frac{\partial E^*}{\partial P_x} = x^*$ AND $\frac{\partial E^*}{\partial P_y} = y^*$

ENVELOPE THEOREM: $\frac{\partial E^*}{\partial P_x} = \frac{\partial L}{\partial P_x}\Big|_{\substack{x=x^* \\ y=y^* \\ \lambda=\lambda^*}}$; $\frac{\partial E^*}{\partial P_y} = \frac{\partial L}{\partial P_y}\Big|_{\substack{x=x^* \\ y=y^* \\ \lambda=\lambda^*}}$

$$L = P_x \cdot x + P_y \cdot y + \lambda\left[\bar{U} - x^{\alpha} y^{1-\alpha}\right]$$

$\frac{\partial L}{\partial P_x} = x \Rightarrow$ EVALUATE THIS AT $x = x^*, y = y^*, \lambda = \lambda^*$ WE HAVE x^*

$\frac{\partial L}{\partial P_y} = y \Rightarrow$ " " " " " y^*

WHEN THE PRICE OF A GOOD INCREASES, TO MAINTAIN THE SAME STANDARD OF LIVING $(\bar{U})$ THE CONSUMER MUST SPEND MORE ON THAT GOOD BY AN AMOUNT THAT IS EQUAL TO THE OPTIMAL COMPENSATED DEMAND OF THAT GOOD.

QUESTION 2, PART a)

$$\max_{x,y} \ (x+\alpha)\cdot(y+\beta) \quad \text{s.t.} \quad p\cdot x + q\cdot y \leq M$$
$$x \geq 0, \ y \geq 0$$

WITH $\alpha, \beta, M > 0$ AND $\alpha p > \beta q$

PART i) K-T CONDITIONS:

$$L = (x+\alpha)\cdot(y+\beta) + \lambda\left[M - p\cdot x - q\cdot y\right]$$

$$\frac{\partial L}{\partial x} \leq 0 \qquad x \geq 0 \qquad x\cdot\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial y} \leq 0 \qquad y \geq 0 \qquad y\cdot\frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial \lambda} \geq 0 \qquad \lambda \geq 0 \qquad \lambda\cdot\frac{\partial L}{\partial \lambda} = 0$$

USE THEOREM 1 LECTURE 4.

SLATER'S CONDITION HOLDS BECAUSE $M > 0$, SO K-T CONDITIONS WELL DEFINED. THEY ARE NECESSARY.

SUFFICIENT? OBJECTIVE FUNCTION AND CONSTRAINT NEEDS TO BE CONCAVE. CONSTRAINT IS LINEAR AND SO CONCAVE. OBJECTIVE FUNCTION IS NOT CONCAVE. ITS HESSIAN IS: $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

SO K-T CONDITIONS ARE ONLY NECESSARY.

PART ii): IF THE CONSTRAINT DOES NOT BIND $p\cdot x + q\cdot y < M$ AT THE SOLUTION. SINCE OBJECTIVE FUNCTION IS INCREASING IN BOTH x AND y, BY SLIGHTLY INCREASIN x AND y (AND STILL SATISFYING THE CONSTRAINT) THE VALUE OF THE OBJECTIVE FUNCTION WILL INCREASE. SINCE WE ARE LOOKING FOR A MAXIMUM OF THE OBJECTIVE FUNCTION, THE CONSTRAINT MUST BIND AT THE SOLUTION

PART iii) CONSTRAINT BINDS AND SO $\lambda > 0$
MOREOVER $x, y > 0$.

INTERIOR SOLUTION:

$$\frac{\partial \mathcal{L}}{\partial x} = y + \beta - p\lambda = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial y} = x + \alpha - q\lambda = 0 \qquad (2)$$

DIVIDE (1) AND (2)

$$\frac{y+\beta}{x+\alpha} = \frac{p}{q} \Rightarrow y = \frac{p(x+\alpha)}{q} - \beta \qquad (3)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - p \cdot x - qy = 0 \qquad (4)$$

USE (3) INTO (4): $M - p \cdot x - q\left(\frac{p}{q}(x+\alpha) - \beta\right) = 0$ SOLVE FOR x:

$$M - 2px - p\alpha + \beta q = 0$$

$$\Rightarrow x^* = \frac{M - p\alpha + \beta q}{2p} \qquad (5)$$

$x^* > 0$ IF $M > p\alpha - \beta q$ INCOME MUST BE SUFFICIENTLY LARGE.

USE (5) INTO (3) TO FIND y:

$$y = \frac{p}{q}\left(\frac{M - p\alpha + \beta q + 2\alpha p}{2p}\right) - \beta$$

$$\Rightarrow y = \frac{M - p\alpha + \beta q + 2\alpha p - 2q\beta}{2q}$$

$$\Rightarrow y^* = \frac{M + \alpha p - \beta q}{2q} > 0$$ SINCE $\alpha p > \beta q$ BY ASSUMPTION

FIND λ^*:

$$\lambda^* = \frac{x^* + \alpha}{q} \qquad \text{FROM EQUATION (2) FOR EXAMPLE}$$

$$\Rightarrow \lambda^* = \frac{M - \alpha P + \beta q + 2pq\alpha}{pq}$$

$$\Rightarrow \lambda^* = \frac{M - (\alpha p - \beta q) + 2pq\alpha}{p \cdot q} > 0 \quad \text{IF } M > \alpha p - \beta q \quad \lambda^* > 0 \text{ FOR SURE SINCE } x^* > 0.$$

K-T CONDITIONS ALL SATISFIED IF $M > \alpha p - \beta q$.

PART b) : MARKET DEMAND $Q_t^D = \alpha - \beta P_{t-1}$

MARKET SUPPLY $Q_t^S = \delta + \varphi P_t$

WITH $\alpha > 0$, $\beta > 0$; $\delta < 0$ AND $\varphi > 0$

PART i) FIRST ORDER DIFFERENCE EQUATION FOR P:

MKT EQUILIBRIUM $\Rightarrow Q_t^D = Q_t^S$

$$\alpha - \beta P_{t-1} = \delta + \varphi P_t$$

$$\Rightarrow P_t = -\left(\frac{\beta}{\varphi}\right) P_{t-1} + \left(\frac{\alpha - \delta}{\varphi}\right)$$

PART ii) PARTICULAR SOLUTION : $P_t = P_{t-1} = P$

$$P = \frac{-\beta}{\varphi} P + \frac{\alpha - \delta}{\varphi}$$

$$P + \frac{\beta}{\varphi} P = \frac{\alpha - \delta}{\varphi}$$

$$\Rightarrow \bar{P} = \frac{\alpha - \delta}{\beta + \varphi} \qquad \text{NOTICE THAT } \bar{P} > 0.$$

COMPLEMENTARY SOLUTION:

HOMOGENEOUS EQUATION: $P_t = \left(\frac{-\beta}{\varphi}\right) P_{t-1}$

SOLUTION $P_t = \left(\frac{-\beta}{\varphi}\right)^t \cdot A$

WHERE A IS AN ARBITRARY CONSTANT.

GENERAL SOLUTION:

$$P_t = \left(\frac{-\beta}{\varphi}\right)^t \cdot A + \frac{\alpha - \delta}{\beta + \varphi}$$

SINCE WE DO NOT KNOW THE INITIAL CONDITION FOR THE PRICE LEVEL WE CANNOT PIN DOWN A UNIQUE VALUE FOR THE CONSTANT A AND THEREFORE THE SOLUTION IS NOT UNIQUE.

PART iii) THE GENERAL SOLUTION FOR P_t CONVERGES TO THE STEADY STATE IF $\lim_{t \to \infty} \left(-\frac{\beta}{\varphi}\right)^t = 0$

THAT IS TRUE IF $\left|\frac{\beta}{\varphi}\right| < 1$. IN THIS CASE WE NEED $|\beta| < |\varphi|$.

THIS MEANS THAT THE SLOPE OF THE DEMAND SHOULD BE LOWER THAN THE SLOPE OF THE SUPPLY (φ) IN ABSOLUTE VALUE.

SINCE WE HAVE $-\beta$, $\frac{-\beta}{\varphi}$ IS NEGATIVE AND IF $|\beta| < |\varphi|$ IS ALSO LESS THAN -1. WE HAVE THEN CONVERGENCE BUT CONVERGENCE IS NOT MONOTONIC.