EC351 Mathematical Economics

Preliminaries

(a) *Contact Details*

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(b) *Course Preamble*

This course aims to provide students with additional mathematical tools to enable them to consider a more interesting set of economic models.

(c) *Course Structure*

The course falls naturally into two sections:

i) Review and extension of optimization techniques: *Lectures* 1-4

ii) Tools for studying dynamic economic problems: *Lectures* 5-10.

(d) *Housekeeping*

i) Reading List: An outline of the course, containing a more detailed list of lecture topics and recommended reading is available in the course materials repository. The main textbook we will use is: *Mathematics for Economics* Hoy (et. al), 2nd Edition, MIT Press, 2001.

ii) Lecture Notes: All lecture notes will be made available from the course materials repository - either before or after the lectures have been delivered.

iii) Examinations: Each student will be assessed by a mid-term test and by means of a 2-hour examination to be taken in May/June

Lecture 1: Convex/Concave Functions and Unconstrained Optimization Theory

Introduction

(a) *Lecture Outline*:

(1.1) Concavity/Convexity of Functions.

(1.2) Quasiconcavity and Quasiconvexity

(1.3) Unconstrained Optimization Theory.

(b) *Essential Reading*:

- Hoy: Chapters 5.5, 11.4, 11.5, 12.1, 12.2

1.1 Concavity/Convexity of Functions

(a) *Functions of One Variable*

- Consider a function $y = f(x)$ which is twice continuously differentiable.
- e.g. $f(x) = x^6$. Then $f'(x) = 6x^5$ and $f''(x) = 30x^4$.
- First derivative function $f'(x)$ measures the rate of change of the function f.
- Second-derivative function $f''(x)$ measures the rate of change of the first derivative $f'(x)$. Therefore it determines the curvature of the function.
- i.e. with a given infinitesimal increase in the independent variable x from a point $x = x_0$:

  i. First derivative $f'(x_0)$:

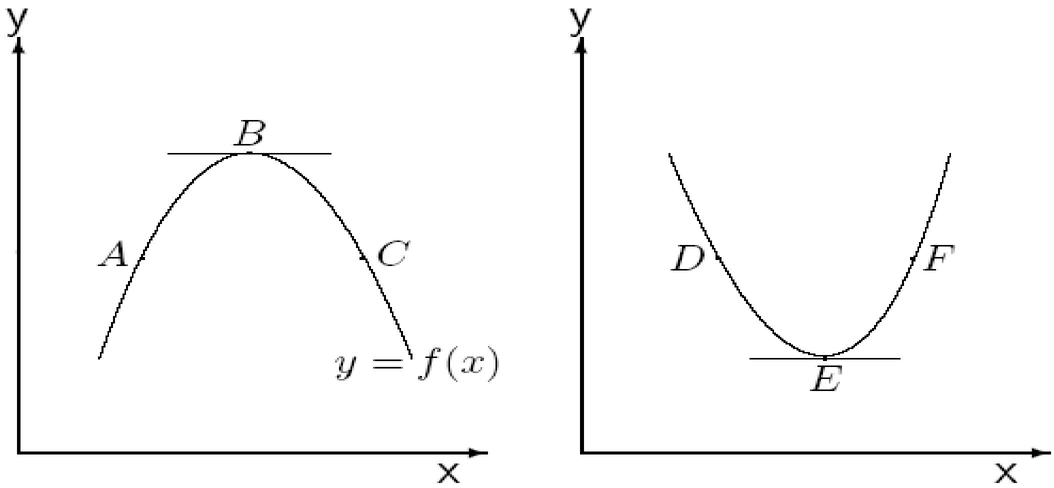
     $f'(x_0) > 0$ the value of the function tends to increase.

     $f'(x_0) < 0$ the value of the function tends to decrease.

  ii. Second derivative $f''(x_0)$:

     $f''(x_0) > 0$ the slope of the function tends to increase.

     $f''(x_0) < 0$ the slope of the function tends to decrease.



(a) $f''(x) < 0$ (b) $f''(x) > 0$

Figure 1: Second Derivative and the curvature of a function

- Figure 1(a) depicts a strictly concave function. When $f''(x) < 0$ the slope steadily decreases as x increases. i.e.
  i. At point A the slope is positive $f'(x) > 0$.
  ii. At point B the slope is zero $f'(x) = 0$.
  iii. At point C the slope is negative $f'(x) < 0$.
- Figure 1(b) depicts a strictly convex function. When $f''(x) > 0$ the slope steadily increases as x increases.
- Geometrically we can pick any pair of points on the curve and join them by a straight line. This line segment must lie entirely below (above) the curve except at these points for a strictly concave (convex) function.
- If the line segment is allowed to lie either below the curve or along the curve then we have a concave function.

Theorem 1 *A twice differentiable function $f(x)$ is strictly concave (convex) if $f''(x) < 0$ ($f''(x) > 0$) except possibly at a single point.*

Theorem 2 *A twice differentiable function $f(x)$ is concave (convex) if $f''(x) \leq 0$ ($f''(x) \geq 0$) on all points of its domain.*

- Example: Suppose $f(x) = x^6$

  $f'(x) = 6x^5$

  $f''(x) = 30x^4 \geq 0$ for every $x \in \mathbb{R}$

  Since $f''(x) = 0$ at a single point i.e. $x = 0$, the function $f(x) = x^6$ is a strictly convex function.

(b) *Functions of n-variables*

- Suppose we have a twice continuously differentiable function $y = f(x)$, $x \in \mathbb{R}^n$.

- Here we need to construct a Hessian matrix composed of all second-order derivatives with the second-order direct partials $\frac{\partial^2 f}{\partial x_i^2}$ (or f_{ii}) on the principal diagonal and the second-order cross partials $\frac{\partial^2 f}{\partial x_i \partial x_j}$ with $i \neq j$ (or f_{ij}) off the principal diagonal.

$$H = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

Theorem 3 *Let H be the Hessian matrix associated with a twice continuously differentiable function $y = f(x)$, $x \in \mathbb{R}^n$. It follows that:*

1. *H is positive definite and so f is strictly convex, if and only if its leading principal minors are positive:*
   $| H_1 |> 0$, $| H_2 |> 0$, $| H_3 |> 0$,..., $| H_n |=| H |> 0 \; \forall x \in \mathbb{R}^n$.

2. *H is negative definite and so f is strictly concave, if and only if its leading principal minors alternate is sign beginning with a negative value:*
   $| H_1 |< 0$, $| H_2 |> 0$, $| H_3 |< 0$,..., $| H_n |=| H |> 0(< 0)$ *if* n *is even (odd)* $\forall x \in \mathbb{R}^n$.

3. *H is positive semidefinite and so f is (weakly) convex, if and only if all its principal minors are positive or zero:*
   $| H_1^* |\geq 0$, $| H_2^* |\geq 0$, $| H_3^* |\geq 0$,..., $| H_n^* |=| H |\geq 0 \; \forall x \in \mathbb{R}^n$.

4. *H is negative semidefinite and so f is concave, if and only if all its principle minors alternate in sign beginning with a negative or zero value.*
   $| H_1^* |\leq 0$, $| H_2^* |\geq 0$, $| H_3^* |\leq 0$,..., $| H_n^* |=| H |\geq 0(\leq 0)$ *if* n *is even (odd)* $\forall x \in \mathbb{R}^n$.

- Example 1:

  Suppose

$$y = 2x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 8$$

  Then the first-order partial derivatives are

$$\begin{aligned} f_1 &= 4x_1 + x_2 + x_3 \\ f_2 &= x_1 + 8x_2 \\ f_3 &= x_1 + 2x_3 \end{aligned}$$

  and the second-order partial derivatives are

$$| H |= \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

  whose leading principal minors are all positive

$$\begin{aligned} | H_1 | &= f_{11} = 4 \\ | H_2 | &= f_{11}f_{22} - f_{21}f_{12} = 32 - 1 = 31 \\ | H_3 | &= 4 \mid 16 - 0 \mid -1 \mid 2 - 0 \mid +1 \mid 0 - 8 \mid = 54 \end{aligned}$$

  Therefore the function is strictly convex.

- A linear function is a (weakly) concave function and a (weakly) convex function.

- The sum of two concave (convex) functions $f + g$ will also be a concave (convex) function z. If either f and/or g are strictly concave then z will be strictly concave.

1.2 Quasiconcavity and Quasiconvexity

- All functions that are concave (convex) are also quasiconcave (quasiconvex).
- However functions that are quasiconcave (quasiconvex) are not necessarily concave (convex).
- Generally speaking a quasiconcave (quasiconvex) function that is not also concave (convex) has a graph roughly shaped like a bell (inverted bell) or a portion thereof.
- To test for quasiconcavity (quasiconvexity) we need to construct a bordered Hessian matrix.
- A border Hessian matrix $\overline{H}$ is comprised by taking the Hessian matrix and adding the first derivatives of the function preceded by a zero as a first column and a first row.

$$\overline{H} = \begin{bmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

- We state two conditions for quasiconcavity (quasiconvexity): one is necessary and the other is sufficient on a domain $\mathbb{R}^n_{++}$

Theorem 4 *Suppose that f is a function defined on $\mathbb{R}^n$ and that f has continuous first- and second-order partial derivatives.*

(i) For f to be quasiconcave on the nonnegative orthant, it is necessary that:
$|\overline{H}_1| \leq 0$, $|\overline{H}_2| \geq 0$, $|\overline{H}_3| \leq 0$,....$|\overline{H}_n| \geq 0$ (≤ 0) *if n is even (odd);*

(ii) For f to be quasiconvex on the nonnegative orthant, it is necessary that:
If $|\overline{H}_1| \leq 0$, $|\overline{H}_2| \leq 0$, $|\overline{H}_3| \leq 0$,....$|\overline{H}_n| \leq 0$;

wherever the partial derivatives are evaluated in the nonnegative orthant.

Theorem 5 *Suppose that f is a function defined on $\mathbb{R}^n$ and that f has continuous first- and second-order partial derivatives.*

(i) A sufficient condition for f to be strictly quasiconcave on the nonnegative orthant is that:
If $|\overline{H}_1| < 0$, $|\overline{H}_2| > 0$, $|\overline{H}_3| < 0$,....$|\overline{H}_n| > 0$ (< 0) *if n is even (odd);*

(ii) A sufficient condition for f to be strictly quasiconvex on the nonnegative orthant is that:
If $|\overline{H}_1| < 0$, $|\overline{H}_2| < 0$, $|\overline{H}_3| < 0$,....$|\overline{H}_n| < 0$;

wherever the partial derivatives are evaluated in the nonnegative orthant.

- Example: $y = x_1 x_2^2$ defined on $\mathbb{R}^2_{++}$ where $x_1, x_2 > 0$.

  The partial derivatives of this function are:

$$\begin{aligned} f_1 &= x_2^2 \\ f_2 &= 2x_1x_2 \\ f_{11} &= 0 \\ f_{22} &= 2x_1 \\ f_{21} = f_{12} &= 2x_2 \end{aligned}$$

  The Bordered Hessian is therefore:

$$\mid \overline{H} \mid = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} 0 & x_2^2 & 2x_1x_2 \\ x_2^2 & 0 & 2x_2 \\ 2x_1x_2 & 2x_2 & 2x_1 \end{vmatrix}$$

  whose leading principal minors are

$$\begin{aligned} \mid \overline{H}_1 \mid &= -\left(x_2^4\right) < 0 \\ \mid \overline{H}_2 \mid &= -x_2^2\left(2x_1x_2^2 - 4x_1x_2^2\right) + 2x_1x_2\left(x_2^2 2x_2\right) \\ &= 6x_1x_2^4 > 0 \end{aligned}$$

  Therefore the function is strictly quasiconcave.

Unconstrained Optimization Theory

- Suppose there are n choice variables, such that the objective function is $y = f(x_1, x_2, .., x_n)$.

- First-order <u>necessary</u> conditions for a local maximum or a local minimum is

$$f_1 = f_2 = .., f_n = 0 \tag{1}$$

Therefore all first-order partial derivatives are required to be zero.

- From the first-order conditions (1) we can compute the stationary values for $x_1^*, ...x_n^*$.

- Second-order <u>sufficient</u> conditions for a local maximum or local minimum are obtained from the Hessian matrix.

(a) $\mid H \mid$ has to be negative definite at $x_1^*, ...x_n^*$ for a local maximum.

Therefore the leading principle minors of the Hessian alternate in sign beginning with a negative value. $\mid H_1 \mid < 0$, $\mid H_2 \mid > 0$, $\mid H_3 \mid < 0$,...

(b) $\mid H \mid$ has to be positive definite at $x_1^*, ...x_n^*$ for a local minimum.

Therefore the leading principle minors of the Hessian must all have a positive sign. $\mid H_1 \mid > 0$, $\mid H_2 \mid > 0$, $\mid H_3 \mid > 0$,...

- Example 1: Suppose a firm produces two products under perfect competition (therefore the price of both products are exogenous).

  The firm's revenue and cost functions are:

$$R = P_1 Q_1 + P_2 Q_2$$
$$C = 2Q_1^2 + Q_1 Q_2 + 2Q_2^2$$

  Therefore the firm's profit function is:

$$\Pi = R - C = P_1 Q_1 + P_2 Q_2 - 2Q_1^2 - Q_1 Q_2 - 2Q_2^2$$

  Nec:

$$\Pi_1 \equiv \frac{\partial \Pi}{\partial Q_1} = P_1 - 4Q_1 - Q_2 = 0$$
$$\Pi_2 \equiv \frac{\partial \Pi}{\partial Q_2} = P_2 - Q_1 - 4Q_2 = 0$$

  Solving for Q_1^* and Q_2^* yields:

$$Q_1^* = \frac{4P_1 - P_2}{15}$$
$$Q_2^* = \frac{4P_2 - P_1}{15}$$

  Suff:

$$| H |= \begin{vmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{vmatrix} = \begin{vmatrix} -4 & -1 \\ -1 & -4 \end{vmatrix}$$

  Therefore the Hessian is negative definite since $| H_1 |= -4 < 0$ and $| H_2 |= 16 - 1 = 15 > 0$.

  Therefore we have a local maximum at Q_1^* and Q_2^*.

  Note however that the signs of the Hessian are independent of where they are evaluated. Thus the Hessian is negative definite everywhere. Therefore the objective function must be strictly concave and the maximum profit is actually a unique global maximum.

- Example 2:

$$y = x_1^3 + x_2^3 - 3x_1x_2$$

<u>Nec</u>:

$$f_1 = 3x_1^2 - 3x_2 = 0$$
$$f_2 = 3x_2^2 - 3x_1 = 0$$

Solving for x_1^* and x_2^* we get:

$$x_1^2 = x_2$$
$$x_2^2 = x_1$$

Therefore two stationary points exist: $x_1^* = x_2^* = 1$ and $x_1^* = x_2^* = 0$.

<u>Suff</u>:

$$\mid H \mid = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} 6x_1^* & -3 \\ -3 & 6x_2^* \end{vmatrix}$$

At the stationary point $x_1^* = x_2^* = 0$, $\mid H_1 \mid = 0$ which prevents us from identifying either a maximum or a minimum.

At the stationary point $x_1^* = x_2^* = 1$, the Hessian is positive definite since $\mid H_1 \mid = 6 > 0$ and $\mid H_2 \mid = 36 - 9 = 27 > 0$.

Therefore we have a local minimum at $x_1^* = x_2^* = 1$.