Lecture 10: Optimal Control Theory

Introduction

(a) *Lecture Outline*:

(10.1) Introduction to Dynamic Optimization.

(10.2) The Hamiltonian Function and the Maximum Principle.

(10.3) Example 1.

(10.4) Example 2: Investment Model.

(b) *Essential Reading*:

- Hoy: Chapters 25.1

10.1 Introduction to Dynamic Optimization

(a) So far in this course we have considered solving static optimization problems. Now given our knowledge of differential and difference equations we are able to consider dynamic optimization problems: i.e. optimization over time.

(b) We will study, at an introductory level, optimal control theory, which is a technique for solving dynamics optimization problems in continuous time.

(c) Optimal control theory relies heavily on the Maximum Principle, which is a set of necessary conditions that hold only on optimal paths.

(d) Thus once we know how to apply these necessary conditions, then we can use our knowledge of differential equations to solve dynamic optimization problems.

(e) In this course we are only going to consider dynamic optimization problems under continuous time in a finite time horizon without discounting with a free endpoint.

(f) The general form of the dynamic optimization problem with a finite time horizon and a free endpoint in continuous time models is

$$\max J = \int_0^T f\left[x(t), y(t), t\right] dt$$

subject to

$$\dot{x} = g\left[x(t), y(t), t\right]$$
$$x(0) = x_0 > 0 \quad (given)$$

(g) The term free endpoint means that $x(T)$ is unrestricted and hence is free to be chosen optimally.

(h) J is the value of the functional which is to be maximized.

(i) We denote $x(t)$ to be the state variable and $y(t)$ to be the control variable where $x(t)$ and $y(t)$ are continuous functions of time.

(j) The control variable is the variable directly chosen, which indirectly influences the state variable.

(k) Suppose that a unique solution to the dynamic optimization problem exists. The solution is a path for the control variable $y(t)$. Once this is specified it will automatically determine the path of $x(t)$ through the differential equation for $x(t)$ combined with its given initial condition.

10.2 The Hamiltonian Function and the Maximum Principle

(a) The Hamiltonian Function for dynamic optimization problems is similar to the Lagrange function in static optimization problems.

(b) The Hamiltonian H is defined as:

$$H\left[x(t), y(t), \lambda(t), t\right] = f\left[x(t), y(t), t\right] + \lambda(t) g\left[x(t), y(t), t\right]$$

where $\lambda(t)$ is referred to as the costate variable (similar to lagrange multiplier in static constrained optimization problems).

(c) One can think of the costate variable $\lambda(t)$ as a sequence or path of Lagrange multipliers. Intuitively the costate variable can be interpreted as the shadow price of the state variable $x(t)$.

(d) The Maximum Principle is a set of necessary conditions that hold only on optimal paths.

(e) The optimal solution path for the control variable $y(t)$ must satisfy the following necessary conditions:

(1) The control variable is chosen to maximize H at each point in time. We assume that the H is strictly concave in y, such that we have an interior solution.

$$\frac{\partial H}{\partial y} = 0$$

(2) Paths of the state $x(t)$ and costate $\lambda(t)$ variables are given by the

solution to the following system of differential equations:

$$\dot{\lambda} = -\frac{\partial H}{\partial x}$$
$$\dot{x} = g\left[x(t), y(t), t\right]$$

(f) The necessary conditions (1) and (2) comprise the maximum principle.

(g) To solve the system we also need two boundary conditions:

$$x(0) = x_0$$
$$\lambda(T) = 0$$

where the second boundary condition is referred to as the transversality condition which is a necessary condition for determining the optimal value of $x(T)$ when $x(T)$ is free to be chosen optimally.

(h) The Maximum principle provides the first-order conditions. We will not consider second-order conditions in this course.

10.3 Example 1:

Solve the following optimal control problem:

$$\max \int_0^1 \left(x - y^2\right) dt$$

subject to:

$$\dot{x} = y$$
$$x(0) = 2$$

- Step 1: Identify the control and state variables.

  Here y is the control variable and x is the state variable since we are given a constraint $\dot{x}$ and initial condition for x_0.

- Step 2: Set up the Hamiltonian

$$H = x - y^2 + \lambda y$$

- Step 3: Apply the Maximum Principle

$$\frac{\partial H}{\partial y} = -2y + \lambda = 0$$
$$\Rightarrow y(t) = \frac{\lambda(t)}{2}$$
$$\frac{\partial H}{\partial x} = 1$$

Therefore we have a system of two differential equations:

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -1$$
$$\dot{x} = y = \frac{\lambda(t)}{2}$$

- Step 4: Obtain the Boundary Conditions

$$x(0) = 2, \quad \lambda(1) = 0$$

- Step 5: Solve the system of linear differential equations.

  Note that since the differential equation for $\dot{\lambda}$ does not depend on x we can solve it directly and then substitute the solution into the second differential equation.

$$\dot{\lambda} = -1$$
$$\frac{d\lambda}{dt} = -1$$
$$d\lambda = -dt$$
$$\int d\lambda = \lambda + C_A$$
$$-\int dt = -t - C_B$$
$$\lambda = C_1 - t$$

  where $C_1 = -(C_B + C_A)$, which is an arbitrary constant of integration. The value of C_1 is found by using the boundary condition $\lambda(1) = 0$. Therefore at $t = 1$

$$\lambda(1) = 0 = C_1 - 1$$
$$\Rightarrow C_1 = 1$$

Therefore we have $\lambda = 1 - t$.

$$\dot{x} = \frac{\lambda}{2} = \frac{1-t}{2}$$
$$\frac{dx}{dt} = \frac{1-t}{2}$$
$$dx = \frac{dt(1-t)}{2}$$
$$\int dx = x + C_D$$
$$\int \frac{dt(1-t)}{2} = \frac{t}{2} - \frac{t^2}{4} + C_Z$$
$$x(t) = \frac{t}{2} - \frac{t^2}{4} + C_2$$

The value of C_2 is found using the boundary condition $x(0) = 2$. Therefore at $t = 0$

$$x(0) = 2 = C_2$$
$$\Rightarrow x(t) = \frac{t}{2} - \frac{t^2}{4} + 2$$

Finally

$$y(t) = \frac{\lambda(t)}{2} = \frac{1-t}{2}$$

is the solution path for the control variable. At $t = 0$, $y(0) = 0.5$. it then declines over time and finishes at $t = 1$ with $y(1) = 0$.

Example 2: Investment

Suppose a firm want's to maximize the integral sum of profits over a given interval of time $(0,T)$.

$$\max \int_0^T \left(K - aK^2 - I^2\right) dt$$

subject to:

$$\dot{K} = I - \delta K$$
$$K(0) = K_0$$

- Step 1: Identify the control and state variables:

  Here capital is the state variable and investment is the control variable.

- Step 2: The Hamiltonian is:

$$H = K - aK^2 - I^2 + \lambda \left(I - \delta K\right)$$

- Step 3: Apply the Maximum Principle

$$\frac{\partial H}{\partial I} = -2I + \lambda = 0$$
$$\Rightarrow I(t) = \frac{\lambda(t)}{2}$$

  i.e. at each moment in time the firm invests up to the point where marginal cost (λ) equals marginal benefit ($2I$).

$$\frac{\partial H}{\partial K} = 1 - 2aK - \lambda\delta$$

Therefore we have a system of two differential equations:

$$\dot{\lambda} = -\frac{\partial H}{\partial K} = \lambda\delta + 2aK - 1$$
$$\dot{K} = I - \delta K = \frac{\lambda}{2} - \delta K$$

- Step 4: Obtain the Boundary Conditions

$$K(0) = K_0, \quad \lambda(T) = 0$$

- Step 5: Solve the system of linear differential equations.

  Here we are going to use the substitution method to solve this linear differential system.

  Homogenous solutions:

$$\dot{\lambda} = \lambda\delta + 2aK$$
$$\Rightarrow \ddot{\lambda} = \dot{\lambda}\delta + 2a\dot{K}$$
$$\ddot{\lambda} = \dot{\lambda}\delta + 2a\left[\frac{\lambda}{2} - \delta K\right]$$
$$\ddot{\lambda} = \dot{\lambda}\delta + 2a\left[\frac{\lambda}{2} - \frac{\delta}{2a}\left(\dot{\lambda} - \lambda\delta\right)\right]$$
$$\ddot{\lambda} - \lambda\delta^2 - a\lambda = 0$$

  The roots of the characteristic equation are:

$$r_1, r_2 = \pm\sqrt{\delta^2 + a}$$

  which are real and distinct. Hence we know that the solution to the system of differential equations is:

$$\lambda(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \overline{\lambda}$$
$$K(t) = \frac{r_1 - \delta}{2a} C_1 e^{r_1 t} + \frac{r_2 - \delta}{2a} C_2 e^{r_2 t} + \overline{K}$$

Furthermore since one of these roots is positive and the other root is negative the steady state equilibrium is a saddle-point equilibrium.

Particular solutions: Setting $\dot{\lambda} = \dot{K} = 0$ we obtain:

$$\lambda = \frac{1 - 2aK}{\delta}$$
$$K = \frac{\lambda}{2\delta}$$

Solving this system for λ and K yields the steady state values:

$$\overline{\lambda} = \frac{\delta}{\delta^2 + a}$$
$$\overline{K} = \frac{1}{2(\delta^2 + a)}$$

Now we only have to find the values for the arbitrary constants of integration C_1 and C_2, which are determined by the boundary conditions $K(0) = K_0$ and $\lambda(T) = 0$. Therefore:

$$K_0 = \frac{r_1 - \delta}{2a} C_1 + \frac{r_2 - \delta}{2a} C_2 + \overline{K}$$
$$\lambda(T) = 0 = C_1 e^{r_1 T} + C_2 e^{r_2 T} + \overline{\lambda}$$

Solving these two equations for C_1 and C_2 yields:

$$C_1 = \frac{2a(K_0 - \overline{K}) + (r_2 - \delta)\overline{\lambda} e^{-r_2 T}}{(r_1 - \delta) - (r_2 - \delta) e^{(r_1 - r_2)T}}$$

$$C_2 = \frac{-2a(K_0 - \overline{K}) e^{(r_1 - r_2)T} - \overline{\lambda}(r_1 - \delta) e^{-r_2 T}}{(r_1 - \delta) - (r_2 - \delta) e^{(r_1 - r_2)T}}$$

Therefore the optimal path of investment is:

$$I^*(t) = \frac{\lambda^*(t)}{2}$$

where λ^* denotes the solution for $\lambda(t)$. We can have two possible solution paths depending on the boundary condition $K(0) = K_0$.

If $K_0 < \overline{K}$ then $I(t)$ starts high and declines monotonically to zero at time T.

If $K_0 > \overline{K}$ then $I(t)$ remains negative from zero to T (i.e. disinvestment).