Lecture 9: Simultaneous Systems of Differential Equations

Introduction

(a) *Lecture Outline*:

(9.1) Linear, Autonomous Differential Equation Systems.

(9.2) The Substitution Method.

(9.3) The Direct Method.

(9.4) Stability Analysis.

(b) *Essential Reading*:

- Hoy: Chapters 24.1, 24.2

9.1 Linear, Autonomous, Differential Equation Systems

(a) So far in this course we have focused specifically at learning the techniques required to solve a single differential equation.

(b) However, in economics it is common to be faced with problems which require two or more variables to be determined simultaneously.

(c) This lecture, we extend the techniques for solving single differential equations, to solve systems of autonomous differential equations.

(d) In this course our focus will be only on solving differential equation systems (and thus due to time constraints we will not consider the solution of systems of difference equations).

(e) In addition we will restrict our attention to solving only linear, autonomous, differential equation systems (and thus not consider the solution of systems of non-linear differential equations)

(f) There are two techniques commonly used to solve linear differential equation systems:

1. **Substitution Method**: which is suited only for solving a differential equation system consisting of exactly two linear differential equations.
2. **Direct Method**: which can be used to solve a linear differential equation system with more than two equations.

9.2 Substitution Method

(a) A linear system, of two autonomous differential equations is expressed as:

$$\dot{y}_1 = a_{11}y_1 + a_{12}y_2 + b_1$$
$$\dot{y}_2 = a_{21}y_1 + a_{22}y_2 + b_2$$

(b) The above equations must be solved simultaneously because $\dot{y}_1$ depends on the solution for y_2 and $\dot{y}_2$ depends on the solution for y_1.

(c) Just like solving a single differential equation, we separate the problem of finding the solutions into two parts:

$$y_1 = y_1^h + y_1^p$$
$$y_2 = y_2^h + y_2^p$$

where y^h is the homogenous solution and y^p is the particular solution.

(d) **The General Solution to the Homogenous Forms**

- To find the homogenous solutions we set the constants b_1 and b_2 equal to zero, to obtain:

$$\dot{y}_1 = a_{11}y_1 + a_{12}y_2 \tag{1}$$

$$\dot{y}_2 = a_{21}y_1 + a_{22}y_2 \tag{2}$$

- To solve equations (1) and (2) we want convert this system into a single second-order differential equation and then we can use the techniques set-out in Lecture 8 to easily solve a linear, autonomous, second-order differential equation!

- Step 1: Differentiate equation (1) to obtain:

$$\ddot{y}_1 = a_{11}\dot{y}_1 + a_{12}\dot{y}_2 \tag{3}$$

which gives us a second-order differential equation.

- Step 2: Use equation (2) to substitute out $\dot{y}_2$ from equation (3).

$$\ddot{y}_1 = a_{11}\dot{y}_1 + a_{12}(a_{21}y_1 + a_{22}y_2) \tag{4}$$

- Step 3: Re-arrange equation (1) in terms of y_2

$$y_2 = \frac{\dot{y}_1 - a_{11}y_1}{a_{12}} \tag{5}$$

- Step 4: Use equation (5) to substitute out y_2 from equation (4)

$$\ddot{y}_1 = a_{11}\dot{y}_1 + a_{12}\left(a_{21}y_1 + a_{22}\frac{\dot{y}_1 - a_{11}y_1}{a_{12}}\right)$$
$$\Rightarrow \ddot{y}_1 - (a_{11} + a_{22})\dot{y}_1 + (a_{11}a_{22} - a_{12}a_{21})y_1 = 0$$

which is a linear, homogenous, second-order differential equation!

- Therefore we can use the solution techniques for linear, autonomous second-order differential equations to find the homogenous forms of this two equation system.

$$\ddot{y} + a_1\dot{y} + a_2 y = 0$$

where

$$a_1 \equiv -(a_{11} + a_{22})$$
$$a_2 \equiv (a_{11}a_{22} - a_{12}a_{21})$$
$$\ddot{y} \equiv \ddot{y}_1$$
$$\dot{y} \equiv \dot{y}_1$$

The characteristic equation is:

$$r^2 + a_1 r + a_2 = 0$$

and the roots are given by:

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

If the roots are real and distinct then we know from Lecture 8 that the solution for y_1^h must be:

$$y_1^h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{6}$$

To find the solution for $y_2^h(t)$ we take the derivative of equation (6) which yields:

$$\dot{y}_1 = r_1 C_1 e^{r_1 t} + r_2 C_2 e^{r_2 t}$$

and substitute this into equation (5) to get:

$$y_2^h(t) = \frac{r_1 - a_{11}}{a_{12}} C_1 e^{r_1 t} + \frac{r_2 - a_{11}}{a_{12}} C_2 e^{r_2 t}$$

Theorem 1: The solutions to the homogenous form of the system of two linear, autonomous first-order differential equations:

$$\dot{y}_1 = a_{11}y_1 + a_{12}y_2 + b_1$$
$$\dot{y}_2 = a_{21}y_1 + a_{22}y_2 + b_2$$

are:

Real and Distinct roots $\left((a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) > 0\right)$

$$y_1^h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$
$$y_2^h(t) = \frac{r_1 - a_{11}}{a_{12}} C_1 e^{r_1 t} + \frac{r_2 - a_{11}}{a_{12}} C_2 e^{r_2 t}$$

Real and Equal roots $\left((a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) = 0\right)$

$$y_1^h(t) = C_1 e^{rt} + C_2 t e^{rt}$$
$$y_2^h(t) = \left[\frac{r - a_{11}}{a_{12}}\left(C_1 + C_2 t\right) + \frac{C_2}{a_{12}}\right] e^{rt}$$

where

$$r_1, r_2 = \frac{a_{11} + a_{22}}{2} \pm \frac{1}{2}\sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}$$

Complex Roots $\left((a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) < 0\right)$

$$y_1^h(t) = e^{ht}\left[A_1 \cos(vt) + A_2 \sin(vt)\right]$$
$$y_2^h(t) = e^{ht}\left[\frac{(h - a_{11})A_1 + vA_2}{a_{12}} \cos(vt) + \frac{(h - a_{11})A_2 - vA_1}{a_{12}} \sin(vt)\right]$$

where

$$h = \frac{1}{2}(a_{11} + a_{22})$$
$$v = \frac{1}{2}\sqrt{4(a_{11}a_{22} - a_{12}a_{21})^2 - (a_{11} + a_{22})^2}$$

- Example: Solve the following system of homogenous differential equations:

$$\dot{y}_1 = y_1 - 3y_2$$
$$\dot{y}_2 = \frac{1}{4}y_1 + 3y_2$$

(e) **The Particular Solutions**

- As always the particular solutions we are interested in are the steady state solutions.
- The steady-state solution to a system of two differential equations is the pair of values $\overline{y}_1$ and $\overline{y}_2$ at which $\dot{y}_1$ and $\dot{y}_2$ are both equal to zero.

$$\dot{y}_1 = a_{11}y_1 + a_{12}y_2 + b_1$$
$$\dot{y}_2 = a_{21}y_1 + a_{22}y_2 + b_2$$

- Set $\dot{y}_1 = \dot{y}_2 = 0$. This gives a linear system of two equations for two unknowns $\overline{y}_1$ and $\overline{y}_2$.

$$a_{11}\overline{y}_1 + a_{12}\overline{y}_2 + b_1 = 0$$
$$a_{21}\overline{y}_1 + a_{22}\overline{y}_2 + b_2 = 0$$

- Re-arrange these two equations for $\overline{y}_1$ and $\overline{y}_2$ respectively:

$$\overline{y}_1 = -\frac{a_{12}}{a_{11}}\overline{y}_2 - \frac{b_1}{a_{11}} \tag{7}$$

$$\overline{y}_2 = -\frac{a_{21}}{a_{22}}\overline{y}_1 - \frac{b_2}{a_{22}} \tag{8}$$

- Use equation (7) to eliminate $\overline{y}_1$ from equation (8):

$$\overline{y}_2 = -\frac{a_{21}}{a_{22}}\left(-\frac{a_{12}}{a_{11}}\overline{y}_2 - \frac{b_1}{a_{11}}\right) - \frac{b_2}{a_{22}}$$

which after re-arranging yields the steady state solution $\overline{y}_2$

$$\overline{y}_2 = \frac{a_{21}b_1 - a_{11}b_2}{a_{11}a_{22} - a_{12}a_{21}} \tag{9}$$

- Substituting equation (9) into equation (7) yields the steady state solution $\overline{y}_1$

$$\overline{y}_1 = \frac{a_{12}b_2 - a_{22}b_1}{a_{11}a_{22} - a_{12}a_{21}} \tag{10}$$

- Therefore the particular solutions are:

$$y_1^p = \overline{y}_1 = \frac{a_{12}b_2 - a_{22}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$
$$y_2^p = \overline{y}_2 = \frac{a_{21}b_1 - a_{11}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

- Example: Find the particular solutions for:

$$\dot{y}_1 = y_1 - 3y_2 + 1$$
$$\dot{y}_2 = \frac{1}{4}y_1 - y_2 + 2$$

(f) Complete Solution: The complete solution to the system of two linear autonomous differential equations is the sum of the homogenous solutions and the particular solutions. Hoy p.939, provides the complete solutions for the three types of characteristic roots that can occur.

(g) Initial conditions:

- When the complete solution must also satisfy given initial conditions, the values of the constants of integration must be set appropriately.

- Example: Find the constants of integration that make the solutions to the following system of differential equations satisfy the initial conditions $y_1(0) = 1$ and $y_2(0) = 3$.

$$\dot{y}_1 = y_1 - 3y_2 - 5$$
$$\dot{y}_2 = \frac{1}{4}y_1 + 3y_2 - 5$$

The homogenous solutions are:

$$y_1^h(t) = C_1 e^{1.5t} + C_2 e^{2.5t}$$
$$y_2^h(t) = -\frac{1}{6}C_1 e^{1.5t} - \frac{1}{2}C_2 e^{2.5t}$$

The particular solutions are:

$$\overline{y}_1 = 8$$
$$\overline{y}_2 = 1$$

Therefore the complete solutions are:

$$y_1^h(t) = C_1 e^{1.5t} + C_2 e^{2.5t} + 8$$
$$y_2^h(t) = -\frac{1}{6}C_1 e^{1.5t} - \frac{1}{2}C_2 e^{2.5t} + 1$$

Evaluate the complete solutions using the initial conditions at $t = 0$ to obtain:

$$C_1 = -7 - C_2$$
$$C_2 = -\frac{2}{6}C_1 - 4$$

Thus we have two equations for two unknown C_1 and C_2. Solving yields:

$$C_1 = -\frac{9}{2}$$
$$C_2 = -\frac{5}{2}$$

Thus the complete solutions that satisfy the initial conditions become:

$$y_1(t) = -\frac{9}{2}e^{1.5t} - \frac{5}{2}e^{2.5t} + 8$$
$$y_2(t) = -\frac{3}{4}e^{1.5t} + \frac{5}{4}e^{2.5t} + 1$$

9.3 Direct Method

(a) Although the substitution approach works well for systems of two differential equations, it becomes cumbersome for larger systems. The following direct approach to solving a system of linear differential equations circumvents this limitation.

(b) A linear system of n autonomous differential equations is expressed in matrix form as:

$$\dot{y} = Ay + b$$

where A is an $n \times n$ matrix of constant coefficients, b is a vector of n constant terms, y is a vector of n variables and $\dot{y}$ is a vector of n derivatives.

(c) e.g. in the case of a system of $n = 2$ linear, autonomous differential equations, the matrix of coefficients is

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and the vector of constant terms is:

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

(d) As before the solution to the complete system of equations is obtained by adding together the homogenous solutions and the particular solutions.

(e) **Homogenous Solutions**

$$\dot{y} = Ay$$

- We proceed by guessing that the homogenous solution are of the form:

$$y = ke^{rt}$$

where k is an n-dimensional vector of constants and r is a scalar.

- To see if this guess is correct, check that the guessed solution and its first derivative satisfy the differential equation system. The derivative of the proposed solution is:

$$\dot{y} = rke^{rt}$$

- Substitution of these derivatives and the proposed solutions into the original system of equations gives:

$$\begin{aligned} rke^{rt} &= Ake^{rt} \\ \Rightarrow [A - rI]k &= \mathbf{0} \end{aligned} \tag{11}$$

where I is an identity matrix and 0 is the zero vector. We wish to find values of r that solve this equation and thus make our guessed solution correct.

- The characteristic equation is

$$| A - rI |= 0$$

where r is the characteristic roots or eigenvalues of matrix A. A nonzero vector k_1 which is a solution to equation (11) for a particular eigenvalue r_1 is called the eigenvector of the matrix A corresponding to the eigenvalue r_1.

- Therefore if $n = 2$ then:

$$| A - rI | = \begin{vmatrix} a_{11} - r & a_{12} \\ a_{21} & a_{22} - r \end{vmatrix} = 0$$

which after simplifying gives

$$r^2 - (a_{11} + a_{22})r + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

which is of course the same characteristic equation obtained from the substitution method.

- Since $| A - rI |$ is constrained to be singular there is an infinite number of eigenvectors that will satisfy the equation. We can either normalize the equation by choosing a vector whose length is unity $k_1^2 + k_2^2 = 1$ which is called the Euclidian distance condition, or we can simply pick any arbitrary value for one element while maintaining the relationship between elements.

- Example: Solve the following system of homogenous differential equations using the direct method:

$$\dot{y}_1 = y_1 - 3y_2$$
$$\dot{y}_2 = \frac{1}{4}y_1 + 3y_2$$

(f) **The particular solutions**

The steady-state solutions provide the particular solutions we require. Set $\dot{y} = 0$ and thus:

$$A\overline{y} + b = 0$$
$$\rightarrow \overline{y} = -A^{-1}b$$

provided the inverse matrix A^{-1} exists (i.e. the determinant of A must be nonzero).

9.4 Stability Analysis

(a) As always we interested in knowing if the system of differential equations converges to the steady state solutions.

(b) The steady state solution of a system of linear, autonomous differential equations is asymptotically stable if and only if the characteristic roots are negative (the real part is negative in the case of complex-valued roots).

(c) If one of the characteristic roots is positive and the other is negative, the steady-state equilibrium is called a saddle-point equilibrium, which is unstable. However $y_1(t)$ and $y_2(t)$ converge to their steady state solutions if the initial conditions for y_1 and y_2 satisfy the following equation:

$$y_2 = \frac{r_1 - a_{11}}{a_{12}}(y_1 - \overline{y}_1) + \overline{y}_2$$

where r_1 is the negative root and r_2 is the positive root. The locus of points (y_1, y_2) defined by this equation is known as the saddle path.