EC371 Economic Analysis of Asset Prices

Arbitrage: basic ideas

	Assets							
	1	2	3	…	n			
State 1	v_{11}	v_{12}	v_{13}	…	v_{1n}	$v(\boldsymbol{x},\,1)$	q_1	π_1
State 2	v_{21}	v_{22}	v_{23}	…	v_{2n}	$v(\boldsymbol{x},\,2)$	q_2	π_2
State 3	v_{31}	v_{32}	v_{33}	…	v_{3n}	$v(\boldsymbol{x},\,3)$	q_3	π_3
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
State ℓ	$v_{\ell 1}$	$v_{\ell 2}$	$v_{\ell 3}$	…	$v_{\ell n}$	$v(\boldsymbol{x},\,\ell)$	q_ℓ	π_ℓ
Price	p_1	p_2	p_3	…	p_n			

v_{kj}: payoff on asset j if state k occurs.

An **arbitrage portfolio**, $x_1,\, x_2,\, \ldots,\, x_n$, satisfies:

1. *Zero initial outlay*: $p_1x_1 + p_1x_2 + \cdots + p_nx_n = 0$, with not all $x_j = 0$ for $j = 1,\, 2, \ldots,\, n$.
2. *Risk free*: $v(\boldsymbol{x},\, k) \equiv v_{k1}x_1 + v_{k2}x_2 + \cdots + v_{kn}x_n \geqq 0$, for every state $k = 1,\, 2,\, \ldots,\, \ell$,

where $\boldsymbol{x}$ denotes the portfolio as a whole (i.e. the vector with elements $\{x_1,\, x_2,\, \ldots,\, x_n\}$). Thus, if $\boldsymbol{x}$ is an arbitrage portfolio, it involves zero initial outlay and $v(\boldsymbol{x},\, k) \geqq 0$ for every state k.

Arbitrage opportunity: a set of asset prices such that an arbitrage portfolio exists, *and* $v(\boldsymbol{x},\, k) > 0$ for at least one k.

Arbitrage profit: the amount of the payoff from an arbitrage opportunity.

Absence of Arbitrage Opportunities: a set of asset prices such that *either* (a) for every arbitrage portfolio, $v(\boldsymbol{x},\, k) = 0$ in every state, *or* (b) no arbitrage portfolio exists (i.e. for every portfolio requiring zero initial outlay, $v(\boldsymbol{x},\, k) \geqq 0$ for some state(s) and $v(\boldsymbol{x},\, k) < 0$ for some state(s)), but not both.

Arbitrage principle: arbitrage opportunities are absent.

Market equilibrium: a set of asset prices and an allocation of asset holdings across investors such that the demand to hold assets is no greater than the supply available.

Proposition I. The arbitrage principle holds in frictionless asset markets if and only if there exists an investor who prefers more wealth to less and for whom an optimal portfolio can be constructed.

Proposition II. The arbitrage principle is equivalent to the existence of positive state prices, $q_1,\, q_2,\, \ldots,\, q_\ell$ such that: $p_j = q_1v_{1j} + q_2v_{2j} + \cdots + q_\ell v_{\ell j}$, $j = 1,\, 2,\, \ldots,\, n$.

Proposition III (RNVR). The linear pricing rule is equivalent to the existence of:

1. a risk-free rate of return, r_0, with associated discount factor, $\delta \equiv 1/(1 + r_0)$, and
2. probabilities, $\pi_1,\, \pi_2,\, \ldots,\, \pi_\ell$, one for each state, such that $p_j = \delta \mathrm{E}^*[v_j]$, $j = 1,\, 2,\, \ldots,\, n$