EC371 Economic Analysis of Asset Prices

Factor Models and the APT

Single factor model

$$r_j = b_{j0} + b_{j1}F_1 + \varepsilon_j, \qquad j = 1, 2, \ldots, n \tag{1}$$

where r_j is the rate of return on asset (or portfolio) j, F_1 denotes the factor's value, b_{j0} and b_{j1} are parameters, and ε_j denotes an unobserved random error. It is assumed that $\mathrm{E}[\varepsilon_j \mid F_1] = 0$, that is, the expected value of the random error, conditional upon the value of the factor, is zero.

Two factor model

$$r_j = b_{j0} + b_{j1}F_1 + b_{j2}F_2 + \varepsilon_j, \qquad j = 1, 2, \ldots, n \tag{2}$$

with factor loadings b_{j1} and b_{j2}.

Multi-factor model (general case)

$$r_j = b_{j0} + b_{j1}F_1 + b_{j2}F_2 + \cdots + b_{jK}F_K + \varepsilon_j, \qquad j = 1, 2, \ldots, n \tag{3}$$

where $K \ll n$. and $\mathrm{E}[\varepsilon_j \mid F_1] = 0$, $\mathrm{E}[\varepsilon_j \mid F_2] = 0, \ldots, \mathrm{E}[\varepsilon_j \mid F_K] = 0$ for each asset j.

Applying the Arbitrage Principle to a Factor Model, results in the APT:

APT prediction, single factor model:

$$\mu_j - r_0 = \lambda_1 b_{j1}, \qquad j = 1, 2, \ldots, n \tag{4}$$

The weight λ_1 is interpreted as the *risk premium* associated with the factor, that is, the risk premium corresponds to the source of the systematic risk.

APT prediction, two factor model:

$$\mu_j - r_0 = \lambda_1 b_{j1} + \lambda_2 b_{j2}, \qquad j = 1, 2, \ldots, n \tag{5}$$

where λ_1 and λ_2 are risk premia associated with the factors F_1 and F_2, respectively.

APT prediction, *multiple* factor model:

$$\mu_j - r_0 = \lambda_1 b_{j1} + \lambda_2 b_{j2} + \cdots + \lambda_K b_{jK}, \qquad j = 1, 2, \ldots, n \tag{6}$$

APT and CAPM

Suppose that (a) the APT holds in a single factor model; and (b) the factor is the excess rate of return on the market portfolio as defined in the CAPM, so that $F_1 = r_M - r_0$. It follows that

$$r_j - r_0 = b_{j1}(r_M - r_0) + \varepsilon_j, \qquad j = 1, 2, \ldots, n \tag{7}$$

and hence

$$\mu_j - r_0 = b_{j1}(\mu_M - r_0), \qquad j = 1, 2, \ldots, n \tag{8}$$

This is exactly the CAPM prediction if b_{j1} is interpreted as the beta-coefficient.

Note: It is possible for the CAPM and APT to be compatible with one another even if the return on the market portfolio is not one of the factors, indeed even if the factors are not portfolio returns at all. (See *Economics of Financial Markets*, pp. 193–194.)