

EC371 Economic Analysis of Asset Prices

Mean-variance portfolios: a single risky asset

The investor is assumed to select a portfolio of just two assets: risk-free lending (or borrowing) at rate r_0 , and one risky asset defined by its expected rate of return, μ_Z and standard deviation of rate of return, σ_Z . It is assumed throughout that $\mu_Z > r_0$.¹

Note that the risky asset (labelled Z) could itself be a mutual fund — a composite asset of many risky assets. Remember that the second mutual fund theorem (separation theorem) for mean-variance investors allows all assets to be grouped into just two, one of which could be the risk-free asset (see *Economics of Financial Markets*, page 127).

The market trade-off

The market “trade-off” denotes the set of portfolios from which the investor will choose for given r_0 , μ_Z and σ_Z . This is depicted in the space of μ_P and σ_P (expected rate of return and standard deviation of rate of return for the portfolio as a whole). See figure 1. Let q denote the proportion of the total portfolio invested in Z — the choice of a portfolio is then just a choice of q (the value of which then determines μ_P and σ_P).

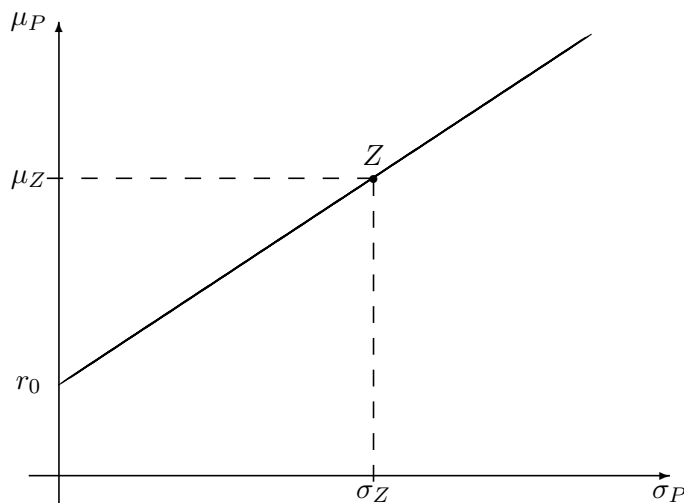


Figure 1: The trade-off between μ_P and σ_P .

The trade-off is often called the set of ‘efficient portfolios’, or in this context, the ‘portfolio frontier’. While all portfolios corresponding to points on, or *below*, the line could be chosen (they are ‘feasible’), it is assumed that the investor always considers μ_P as ‘good’ and σ_P as ‘bad’. Hence, only portfolios on the line will ever be chosen.

¹For a discussion of the peculiar case in which $\mu_Z \leq r_0$, see *Economics of Financial Markets*, pages 125–7.

At the point on the vertical axis denoted by r_0 , the whole portfolio is invested in the risk-free asset, i.e. $q = 0$. At Z , the whole portfolio is invested in the risky asset, i.e. $q = 1$.

Note that for points between r_0 and Z , $0 < q < 1$. For points on the line to the right of Z , $\mu_P > \mu_Z$ and $\sigma_P > \sigma_Z$: the investor is *borrowing* funds to invest in the risky asset, i.e. $q > 1$.

The slope of a straight line is defined simply as the ratio of the vertical distance to the horizontal along the line. Hence, the slope equals $\mu_Z - r_0$ divided by σ_Z , i.e., $(\mu_Z - r_0)/\sigma_Z$.

The investor's objective function

In mean-variance analysis it is assumed that the investor acts to maximize an objective function, $G(\mu_P, \sigma_P^2)$, where $G_1 > 0$: expected return is 'good', and $G_2 < 0$: variability of return is 'bad' (subscripts denote partial differentiation, $G_1 \equiv \partial G/\partial \mu_P$; $G_2 \equiv \partial G/\partial \sigma_P^2$). This note focuses on a special case for which the objective can be written as:

$$G(\mu_P, \sigma_P^2) = \mu_P - \alpha \sigma_P^2, \quad (1)$$

where $\alpha > 0$ is a parameter that expresses the investor's risk preferences: the larger is α , the more averse the investor is to risk (i.e. gives more weight to σ_P^2 in the objective function). Representative indifference curves appear in figure 2.

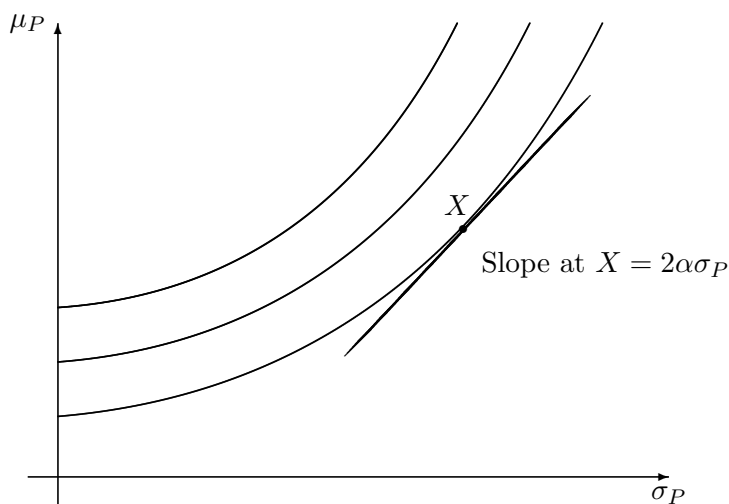


Figure 2: Indifference curves for $G(\mu_P, \sigma_P^2) = \mu_P - \alpha \sigma_P^2$.

By construction an indifference curve is defined for a fixed level of utility, G , say g_0 . Solving for μ_P , the indifference curve can be expressed as $\mu_P = g_0 + \alpha \sigma_P^2$. The slope of the indifference curve is then given by the derivative of μ_P with respect to σ_P , for a fixed level of the objective function $G = g_0$:

$$\left. \frac{d\mu_P}{d\sigma_P} \right|_{G=g_0} = 2\alpha\sigma_P.$$

Notice that the slope depends on σ_P (and α) but not on μ_P — on a vertical line drawn at any given σ_P , the slope of every indifference curve is the same, i.e. $2\alpha\sigma_P$. This result

holds only for the particular objective function assumed in (1): it need not hold for other $G(\mu_P, \sigma_P^2)$ functions.

Notice also that the slope of the indifference curve depends positively on σ_P : the curves are convex from below. (Formally, $d^2\mu_P/d\sigma_P^2 = 2\alpha > 0$.)

For later reference, note that $\sigma_P = q\sigma_Z$. To understand this result, recognise that the risk-free asset contributes nothing to the variability of return ('risk') of the portfolio. The risk, expressed as the standard deviation of portfolio return, is proportional to the standard deviation of the return on the risky asset.² The larger the value of q that is chosen, the greater the portfolio risk.

The optimum portfolio

The investor is assumed to choose q (the proportion of the portfolio in the risky asset) to maximize $G(\mu_P, \sigma_P^2)$ from among the feasible portfolios. In words: the optimum portfolio corresponds to the highest indifference curve that can be reached along the market trade-off of efficient portfolios.

At the optimum portfolio, the trade-off of feasible portfolios is tangential to an indifference curve, i.e. their slopes are equal.³ See figure 3 on page 4.

The slope of the trade-off equals $(\mu_Z - r_0)/\sigma_Z$ while the slope of each indifference curve equals $2\alpha\sigma_P = 2\alpha q\sigma_Z$. Hence, letting q^* denote the optimum value of q :

$$\frac{\mu_Z - r_0}{\sigma_Z} = 2\alpha q^* \sigma_Z \quad \text{so that} \quad q^* = \frac{\mu_Z - r_0}{2\alpha\sigma_Z^2}$$

(From now on, the asterisk superscript will be omitted if no confusion would result, i.e. q is the value at the optimum unless otherwise stated.)

Figure 3 shows the optimum when $0 < q < 1$, i.e. for a lender — an investor who chooses to hold a positive amount of the risk-free asset. It is possible, however, that $q > 1$. This corresponds to a *borrower* — see figure 4, on page 4. For given values of r_0 , μ_Z and σ_Z , whether the investor is a lender or a borrower depends on the value of α , i.e. on the investor's aversion to risk.

²Proof (*optional* for EC371): $\sigma_P^2 \equiv \text{E}[(r_P - \mu_P)^2]$, where $r_P = qr_Z + (1 - q)r_0$, and r_Z is the rate of return on the risky asset. Thus, $\mu_P = q\mu_Z + (1 - q)r_0$, and $r_P - \mu_P = q(r_Z - \mu_Z)$, so that: $\sigma_P^2 = q^2\text{E}[(r_Z - \mu_Z)^2] = q^2\sigma_Z^2$. Taking the positive square root gives $\sigma_P = q\sigma_Z$, as asserted.

³Two caveats are necessary. First, there may be an upper-bound on the borrowing that the investor can undertake at r_0 . It is possible that this upper-bound will be reached for a smaller level of q than at which an indifference curve is tangential to the trade-off.

Second, it is possible that the highest indifference curve is reached, without being tangential to the trade-off, on the vertical axis, at r_0 , for which $q = 0$ (the investor avoids risky assets altogether). This cannot hold for the $G(\mu_P, \sigma_P^2)$ assumed here: the slope of every indifference curve on the vertical axis equals zero. Because the trade-off has a positive slope, every investor will seek to choose $q > 0$, irrespective of the intensity of risk aversion, i.e., $q > 0$ for any value of α (if $\mu_Z > r_0$ as assumed).

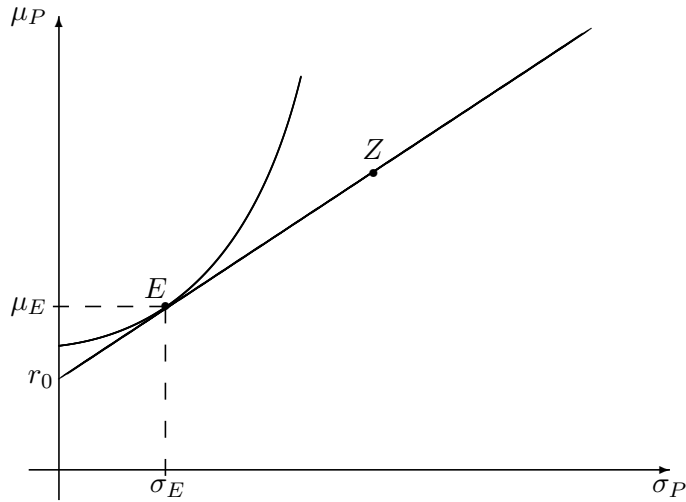


Figure 3: The optimum portfolio: a *lender*, $0 < q \leq 1$.

The optimum portfolio is that which results in the greatest utility (i.e. highest indifference curve) subject to being on the trade-off (the boundary of feasible portfolios). This occurs at a tangency between the trade-off and an indifference curve at E , with expected return μ_E and ‘risk’ σ_E .

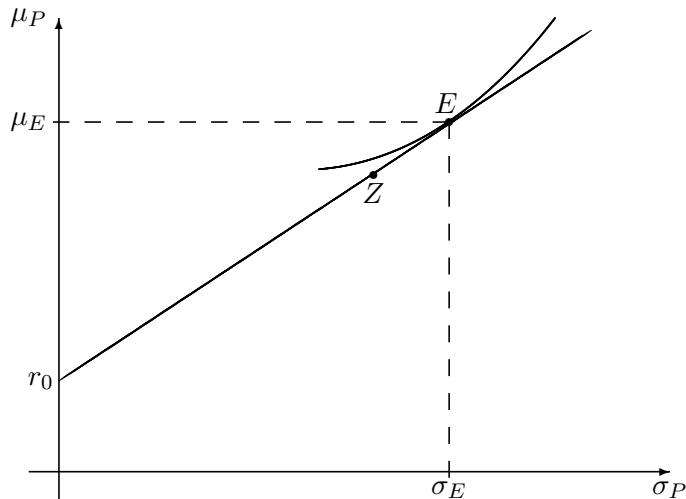


Figure 4: The optimum portfolio: a *borrower*, $q > 1$.

For a borrower, $q > 1$, and E lies to the right of Z . Borrowing is interpreted as ‘negative lending’ because the proportion of the portfolio invested in the risk-free asset is negative, $1 - q < 0$.

An increase in the risk-free rate

Suppose that the risk-free rate increases from r_0 to r'_0 , while μ_Z and σ_Z remain unchanged. See figures 5 (lender) and 6 (borrower) on page 6.

In the figures, the trade-off pivots through the point μ_Z, σ_Z , becoming flatter. The optimum portfolio, changes from E to E' . In each case E' lies to the left of E , the proportion of the portfolio invested in the risky asset falls (the risk-free rate of return has become more attractive), i.e. q falls. The lender becomes better off (higher indifference curve), and the borrower becomes worse off.⁴

The change in q is a result of ‘substitution’ and ‘scale’ (‘income’) effects. It is not true for *every* utility function, $G(\mu_P, \sigma_P^2)$, that q falls in response to a rise in r_0 . But it *does* hold for the special case considered here. To prove this, differentiate q^* with respect to r_0 :

$$\frac{dq^*}{dr_0} = \frac{-1}{2\alpha\sigma_Z^2} < 0.$$

Thus, figures 5 and 6 are consistent with a mean-variance objective function of the form $G(\mu_P, \sigma_P^2) = \mu_P - \alpha\sigma_P^2$.

A caveat and warning

Suppose that the risky asset, Z , is not a single asset but instead is a mutual fund comprising many risky assets. In this case, μ_Z and σ_Z will change *in response to* a change in r_0 . When r_0 increases, the composition of the mutual fund changes in response: μ_Z and σ_Z both increase. See figure 7 on page 7. For construction of the efficient set (trade-off) with many risky assets and a risk-free asset, see *Economics of Financial Markets*, section 5.4.1 pp. 125–127.

⁴It is not *always* the case that an investor who borrows at r_0 will become worse off if the risk-free rate increases. Why?

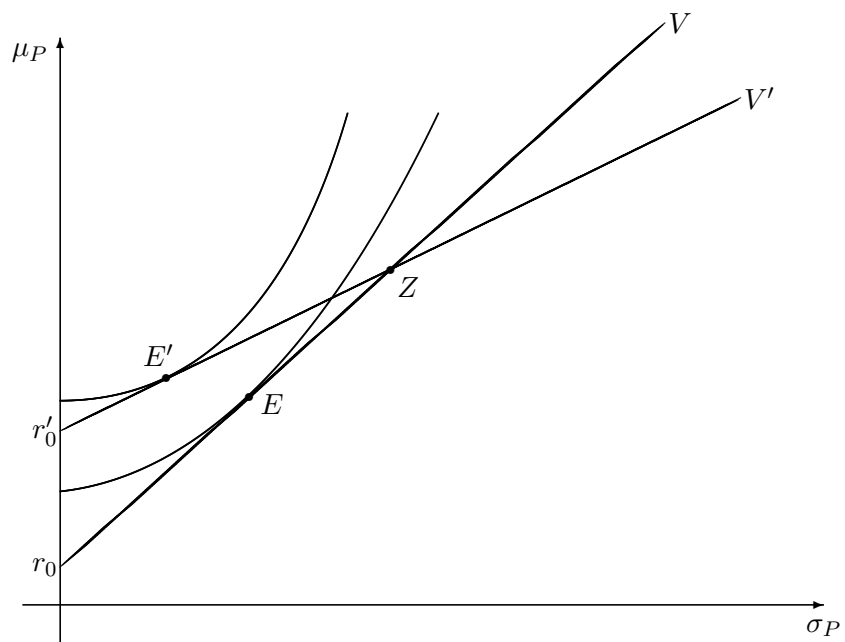


Figure 5: The effect of an increase in the risk-free interest rate: a *lender*.

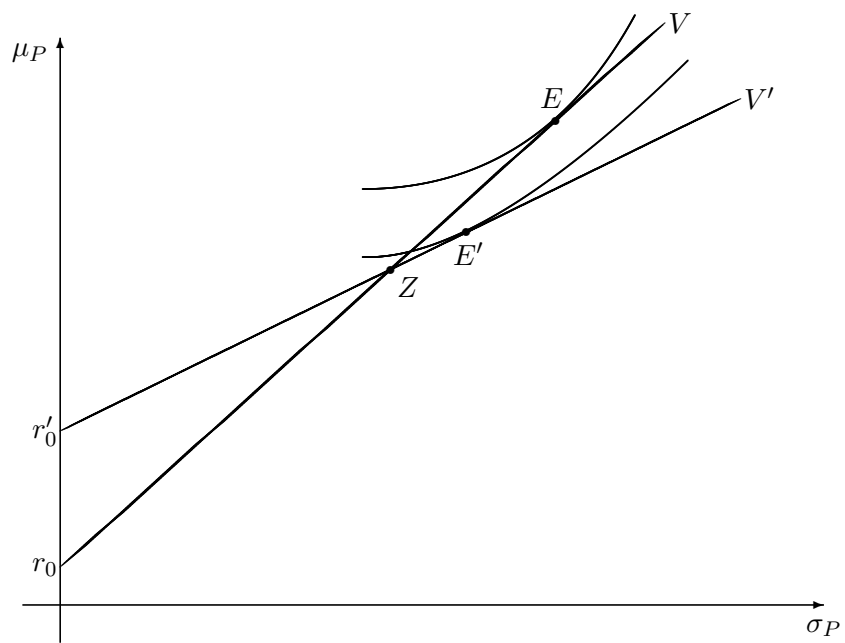


Figure 6: Effect of an increase in the risk-free interest rate: a *borrower*.

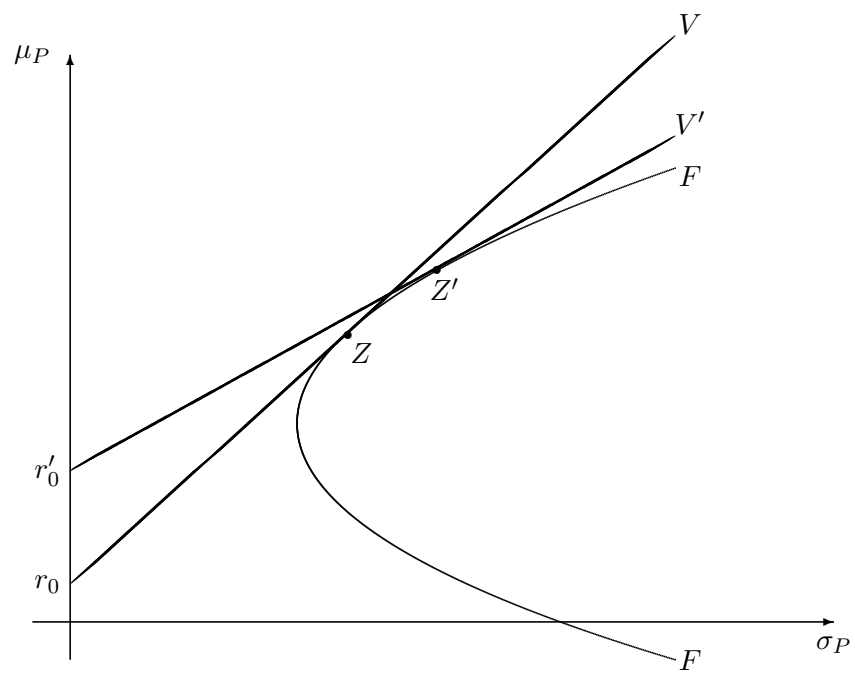


Figure 7: An increase in the risk-free rate with many risky assets.