

EC371 Economic Analysis of Asset Prices

Basic Ideas of Probability and Expectation

The purpose of this note is review some elementary ideas in probability theory, starting from the assumption that ‘probabilities’ (non-negative numbers that sum to 1) have somehow been assigned to ‘events’ (typically the outcome of a random variable – something that is not known in advance, such as the payoff on an investment or a stock price).

Much of financial economics is about the ‘expectations’ of investors. Probability theory is a set of logical principles that helps to make the meaning of ‘expectations’ precise.

Where do the probabilities come from? Who determines the values they take? How are they assigned? This is where economic ideas come in. The view taken in EC371 is that ‘probability’ is a formal expression of ‘degree of belief’. How investors form their beliefs – and whether it is plausible to model beliefs as probabilities – is a main theme of financial economics. This is why an understanding of elementary probability theory is necessary.

1 Expectation

Value of random variable, Y :	y_1	y_2	\dots	y_j	\dots	y_n
Probability:	π_1	π_2	\dots	π_j	\dots	π_n

$$\begin{aligned} \text{Expectation of } Y : \quad E[Y] &= \pi_1 y_1 + \pi_2 y_2 + \pi_3 y_3 + \dots + \pi_n y_n \\ &= \sum_{j=1}^n \pi_j y_j \end{aligned}$$

Alternative notation for expectation: $\mu \equiv E[Y]$. (Include a subscript when it is necessary to identify *which* random variable (r.v.), e.g. $\mu_y \equiv E[Y]$.)

Example:

Suppose that Y denotes the payoff on an investment (measured, say, in dollars).

Value of random variable, Y :	-4	0	8
Probability:	0.2	0.4	0.4

$$\text{Expectation of } Y : \quad E[Y] = 0.2 \times (-4) + 0.4 \times 0 + 0.4 \times 8 = 2.4$$

The expected payoff on the investment is \$2.4.

1.1 Expectation of a function of a random variable

Example: Suppose that the price of the investment (above) is \$2.

Then the *rate of return* on the investment, Z , is its $\frac{\text{payoff minus price}}{\text{price}} = \frac{\text{payoff}}{\text{price}} - 1$:

Rate of return, random variable Z :	-3	-1	3
Probability:	0.2	0.4	0.4

$$\text{Expected rate of return: } E[Z] = 0.2 \times (-3) + 0.4 \times (-1) + 0.4 \times 3 = 0.2$$

The expected rate of return on the investment is 0.2 (i.e., 20%).

Notice that $Z = -1 + Y/2 = a + bY$, where $a = -1$ and $b = 1/2$.

Hence, $E[Z] = -1 + E[Y]/2 = -1 + 2.4/2 = 0.2$. More generally, let $Z = a + bY$. Then:

$$E[Z] = E[a + bY] = a + bE[Y]$$

Important: this relationship holds *only* for linear functions.

2 Variance

$$\begin{aligned} \text{Variance of } Y : \quad \text{var}[Y] &= \pi_1(y_1 - \mu)^2 + \pi_2(y_2 - \mu)^2 + \pi_3(y_3 - \mu)^2 + \cdots + \pi_n(y_n - \mu)^2 \\ &= \sum_{j=1}^n \pi_j(y_j - \mu)^2 \\ &= E[(Y - \mu)^2] \equiv E[(Y - E[Y])^2] \quad (\text{because } \mu \equiv E[Y]) \end{aligned}$$

$$\text{Standard deviation of } Y : \quad \text{s.d.}[Y] = +\sqrt{\text{var}[Y]}$$

Alternative notation: $\sigma^2 \equiv \text{var}[Y]$, and $\sigma \equiv \text{s.d.}[Y]$. (Include a subscript when it is necessary to identify *which* random variable, e.g. $\sigma_y^2 \equiv \text{var}[Y]$, $\sigma_y \equiv \text{s.d.}[Y]$. Sometimes, also: $\sigma_{yy} \equiv \text{var}[Y]$.)

Example:

Value of random variable, Y :	-4	0	8
$(Y - E[Y])^2$:	40.96	5.76	31.36
Probability:	0.2	0.4	0.4

$$\text{Variance of } Y : \quad \text{var}[Y] = 0.2 \times 40.96 + 0.4 \times 5.76 + 0.4 \times 31.36 = 23.04$$

$$\text{Standard deviation of } Y : \quad \text{s.d.}[Y] = +\sqrt{23.04} = 4.8$$

2.1 Variance of a function of a random variable

Example (continued): To calculate the variance of the *rate of return*, $Z = -1 + Y/2$ on the investment, it is possible to first calculate the values of the random variable Z , then to calculate the squared deviations from its expectation, as above. Alternatively:

Let $Z = a + bY$, then:

$$\text{var}[Z] = \text{var}[a + bY] = b^2 \text{var}[Y]$$

(While you should be able to derive this result, you are *not* required to do so for EC371.)

Applying the result to the example:

$$\text{Variance of rate of return, } Z : \text{var}[Z] = \left(\frac{1}{2}\right)^2 \times 23.04 = 5.76$$

$$\text{Standard deviation of rate of return, } Z : \text{s.d.}[Z] = b \times \text{s.d.}[Y] = \left(\frac{1}{2}\right) \times 4.8 = 2.4$$

$$\text{Also, } \text{s.d.}[Z] = +\sqrt{\text{var}[Z]} = +\sqrt{5.76} = 2.4$$

Important: the relationship above holds *only* for linear functions.

3 Two Random Variables

Example:

		Value of r.v. Y		
		-4	0	8
Value of r.v. X :	3	0	0.1	0.4
	2	0.2	0.3	0

The six probabilities, 0, 0.1, 0.4, 0.2, 0.3, 0, correspond to the joint occurrence of both X and Y , so that the probability of the event that “ $X = 2$ and $Y = -4$ ” equals 0.2, $\text{Prob}(X = 2, Y = -4) = 0.2$.

The expectations and variances of X and Y are defined using *marginal* probabilities (summing rows and columns) as follows:

		Value of r.v. Y			
		-4	0	8	
Value of r.v. X :	3	0	0.1	0.4	0.5
	2	0.2	0.3	0	0.5
		0.2	0.4	0.4	1

Thus, for example, $E[X] = 0.5 \times 3 + 0.5 \times 2 = 2.5$,
and $\text{var}[X] = 0.5 \times (3 - 2.5)^2 + 0.5 \times (2 - 2.5)^2 = 0.25$, so that $\text{s.d.}[X] = 0.5$.

More generally, letting $\pi_{ij} \equiv \text{Prob}(X = x_i, Y = y_j)$:

		Value of random variable, Y:						
		y_1	y_2	\dots	y_j	\dots	y_n	
	x_1	π_{11}	π_{12}	\dots	π_{1j}	\dots	π_{1n}	π_{x1}
	x_2	π_{21}	π_{22}	\dots	π_{2j}	\dots	π_{2n}	π_{x2}
				\vdots	\vdots			
Value of random variable, X:	x_i	π_{i1}	π_{i2}	\dots	π_{ij}	\dots	π_{in}	π_{xi}
				\vdots	\vdots			
	x_m	π_{m1}	π_{m2}	\dots	π_{mj}	\dots	π_{mn}	π_{xm}
Marginal probability of Y:		π_{y1}	π_{y2}	\dots	π_{yj}	\dots	π_{yn}	1

Marginal probabilities: for X , sum across the rows, e.g., $\pi_{xi} = \pi_{i1} + \pi_{i2} + \dots + \pi_{in}$; for Y , sum down the columns, e.g., $\pi_{yj} = \pi_{1j} + \pi_{2j} + \dots + \pi_{mj}$.

3.1 Covariance and correlation

Covariance between X and Y : $\text{cov}[X, Y] = E[(X - \mu_x)(Y - \mu_y)]$

It can be shown that: $\text{cov}(X, Y) = E[XY] - \mu_x\mu_y$.

Example (continued):

Value of (XY) :	-12	0	24	-8	0	16
Probability:	0	0.1	0.4	0.2	0.3	0

Thus:

$$E[XY] = 0 \times (-12) + 0.1 \times 0 + 0.4 \times 24 + 0.2 \times (-8) + 0.3 \times 0 + 0 \times 16 = 8.0$$

$$\text{cov}[X, Y] = E[XY] - \mu_x\mu_y = 8.0 - (2.5 \times 2.4) = 2.0$$

Notice that $\text{cov}[Y, Y] = \text{var}[Y] = \sigma_{yy}$.

Alternative notation: $\sigma_{xy} \equiv \text{cov}[X, Y]$.

$$\text{Correlation between } X \text{ and } Y: \rho_{xy} \equiv \text{corr}[X, Y] = \frac{E[(X - \mu_x)(Y - \mu_y)]}{\text{s.d.}[X] \times \text{s.d.}[Y]} = \frac{\sigma_{xy}}{\sigma_x\sigma_y}$$

It can be shown that: $-1 \leq \rho_{xy} \leq +1$, always.

Example (continued):

$$\text{Correlation between } X \text{ and } Y: \text{corr}[X, Y] = \frac{2.0}{0.5 \times 4.8} = 0.83\bar{3}$$

3.2 Functions of two random variables

Suppose that X and Y are the payoffs on two investments. The payoff on a portfolio, W , of these two investments is $W = aX + bY$, where a and b denote the number of units of X and Y , respectively in the portfolio.

The expected payoff on the portfolio is:

$$E[W] = E[aX + bY] = aE[X] + bE[Y]$$

It can be shown that the variance of the payoff is:

$$\text{var}[W] = \text{var}[aX + bY] = a^2\text{var}[X] + b^2\text{var}[Y] + 2ab \cdot \text{cov}[X, Y]$$

More compactly: $\sigma_W^2 = a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\sigma_{xy}$.

Example (continued):

Suppose that a portfolio comprises $a = 2$ units of X and $b = 5$ units of Y . Then:

$$\text{Expected payoff: } E[W] = 2 \times 2.5 + 5 \times 2.4 = 17$$

$$\text{Variance of payoff: } \text{var}[W] = 2^2 \times 23.04 + 5^2 \times 0.25 + 2 \times 2 \times 5 \times 2.0 = 138.41$$

$$\text{Standard deviation: } \text{s.d.}[W] = \sqrt{138.41} \approx 11.765$$

3.3 Conditional probability and expectation

To motivate this section, interpret the outcome of a random variable X as ‘information’. Probabilities (as expressions of beliefs) may then be affected by – are *conditional upon* – the outcome of X . When the value of X becomes known, it may affect the expectation of another random variable, Y .

Suppose that $X = x_i$

Value of Y :	y_1	y_2	\dots	y_j	\dots	y_n
Prob. of Y given $X = x_i$:	$\frac{\pi_{i1}}{\pi_{xi}}$	$\frac{\pi_{i2}}{\pi_{xi}}$	\dots	$\frac{\pi_{ij}}{\pi_{xi}}$	\dots	$\frac{\pi_{in}}{\pi_{xi}}$

Example (continued):

Value of random variable, Y :	-4	0	8
Probability of Y given $X = 3$:	0	0.2	0.8

$$\begin{aligned} \text{Expectation of } Y \text{ given } X = x_i : \quad E[Y | X = x_i] &= \frac{\pi_{i1}}{\pi_{xi}}y_1 + \frac{\pi_{i2}}{\pi_{xi}}y_2 + \dots + \frac{\pi_{in}}{\pi_{xi}}y_n \\ &= \sum_{j=1}^n \frac{\pi_{ij}}{\pi_{xi}}y_j \end{aligned}$$

Example (continued):

Expectation of Y given $X = 3$: $E[Y | X = 3] = 0 \times (-4) + 0.2 \times 0 + 0.8 \times 8 = 6.4$

Also, note that $E[Y | X = 2] = -1.6$.

It is a useful fact (the ‘theorem of iterated expectations’) that: $E[E[Y | X]] = E[Y]$.

Example (continued):

$E[E[Y|X]] = 0.5 \times E[Y|X = 2] + 0.5 \times E[Y|X = 3] = 0.5 \times (-1.6) + 0.5 \times 6.4 = 2.4 = E[Y]$

3.4 Independence of random variables

Two random variables X and Y are defined to be *independent* if:

$$\text{Prob}(X = x_i, Y = y_j) = \text{Prob}(X = x_i) \times \text{Prob}(Y = y_j), \quad \text{for every } i \text{ and } j$$

The independence of random variables is intimately related to conditional expectation and covariance as follows:

$$\text{Independence of } X \text{ and } Y \implies E[Y | X] = 0 \implies \text{cov}(X, Y) = 0.$$

But *not* conversely (i.e., $\text{cov}(X, Y) = 0$ does not imply that X and Y are independent).

4 Sample means, variances and covariances

Be careful to distinguish the *theoretical* concepts described above from their *sample* analogues. *They are different. Do not confuse them.*

Sample statistics can be interpreted *empirical counterparts* of the theoretical means (expectations), variances, covariances. The theory of *statistical inference* (not covered here) studies how, under particular assumptions, the sample statistics can be used to *estimate* theoretical values (which are never observed).

For example, a sample of rates of return on an investment might have been obtained, taking on values such as: 0.05, 0.20, -0.1, 0.15, -0.04, 0.022. The sample mean (average) is:

$$\frac{0.05 + 0.20 - 0.1 + 0.15 - 0.04 + 0.22}{6} = 0.08$$

The relationship between this value, 8%, and the *expectation* (theoretical mean) of the rate of return depends on the model of the process that is assumed to generate the asset’s prices and payoffs – i.e., the subject matter of the economics of finance.

It is important to keep clear that there is a distinction between *theoretical* values and their *empirical* counterparts. *They are different things. Do not confuse them.*
