EC372 Economics of Bond and Derivatives Markets
Nominal Bonds, Real Bonds and Expected Inflation

This note uses zero-coupon (ZC) bonds to explore the so-called 'Fisher relationship' between rates of return and the rate of inflation:

$$\text{nominal rate of return} = \text{real rate of return} + \text{expected rate of inflation}$$

Strictly, as will be shown below, the relationship should be written (the Fisher relationship is an approximation[1]):

$$(1 + \text{nominal rate of return}) = (1 + \text{real rate of return}) \times (1 + \text{expected rate of inflation})$$

1. Expected rate of inflation

Suppose that the prices of goods and services (the 'price level') change by a factor Z_n between today and n years from now. Then the average annual rate of inflation over the coming n years can be defined as the π_n that satisfies:

$$(1+\pi_n)^n = Z_n \quad \text{or} \quad \pi_n = Z_n^{1/n} - 1 \tag{1}$$

For example, if the price level is 1.5 times higher in five years than it is today, then $Z_5 = 1.5$ and the average annual rate of inflation over over the coming five years equals $\pi_5 = (1.5)^{1/5} - 1 \approx 8.45\%$.[2]

As of today, the price level at any date in the future cannot be known for sure. Hence, Z_n should be interpreted as an *expectation* (i.e., based on investors' beliefs – exactly *whose* beliefs will be discussed later).

Bond yields (see *Economics of Financial Markets*, chapter 12, for details)

Two ZC bonds are assumed to exist: *nominal* ZC bonds and *real* ZC bonds, both with a maturity n years from today. For example, suppose that $n = 5$ and the face value of each bond equals $m = \$100$.

Notation (defined below):

	Spot yields	
	Nominal	Real
Nominal ZC bond	y_n	y_n^*
Real ZC bond	$\widetilde{y}_n$	$\widetilde{y}_n^*$

(Summary: * denotes real *yields*; $\widetilde{}$ denotes real *bonds*.)

Nominal ZC bonds:

The nominal spot yield on a nominal ZC bond is its *yield to maturity*, the rate of return that is obtained if the bond is held to maturity. Hence, y_n satisfies:

$$p_n = \frac{m}{(1+y_n)^n} \quad \text{that is:} \quad p_5 = \frac{100}{(1+y_5)^5} \tag{2}$$

[1] The Fisher relationship is an exact equality if rates of return are *continuously compounded* (see *Economics of Financial Markets*, appendix 1.3, pp. 29–31. This note assumes that rates of return are compounded annually.

[2] With continuous compounding, the annual rate of inflation is $\ln(1.5)/5 \approx 8.11\%$.

where p_n is the bond's market price today and $m = \$100$ is its face value.

The *real* spot yield on a nominal ZC bond is its yield to maturity, where the payoff, \$100, is adjusted ('deflated') for changes in the price level of goods and services between today and the maturity date.

Hence, the real payoff on the nominal ZC bond is $m/Z_n = 100/Z_5$ – the nominal \$100 has been 'deflated' by the rise in the price level. Thus, y_n^* is defined to satisfy:

$$p_n = \frac{m/Z_n}{(1+y_n^*)^n} \quad \text{that is:} \quad p_5 = \frac{100/Z_5}{(1+y_5^*)^5} \tag{3}$$

Real ZC bonds:

The *real* spot yield on a real ZC bond is its yield to maturity, the rate of return that is obtained if the bond is held to maturity, omitting the requirement to increase the bond's face value according to changes in the price level – i.e., *as if* the price level remains constant. Hence, $\widetilde{y}_n^*$ satisfies:

$$\widetilde{p}_n = \frac{m}{(1+\widetilde{y}_n^*)^n} \quad \text{that is:} \quad \widetilde{p}_5 = \frac{100}{(1+\widetilde{y}_5^*)^5} \tag{4}$$

The *nominal* spot yield on a real ZC bond is its yield to maturity, where the payoff, \$100, is adjusted ('inflated') for changes in the price level of goods and services between today and the maturity date. Hence, the nominal payoff on the real ZC bond is $m \times Z_n = 100 \times Z_5$ – the nominal \$100 has been 'inflated' by the rise in the price level. Thus, $\widetilde{y}_n$ is defined to satisfy:

$$\widetilde{p}_n = \frac{m \times Z_n}{(1+\widetilde{y}_n)^n} \quad \text{that is:} \quad \widetilde{p}_5 = \frac{100 \times Z_5}{(1+\widetilde{y}_5)^5} \tag{5}$$

Note: in each case it is possible to solve explicitly for the spot yield. For example:

$$y_n = \left(\frac{m}{p_n}\right)^{1/n} - 1 \quad \text{and} \quad \widetilde{y}_n^* = \left(\frac{m}{\widetilde{p}_n}\right)^{1/n} - 1 \tag{6}$$

2. Interest rates and expected inflation

Interpret y_n as the *nominal* rate of return and $\widetilde{y}_n^*$ as the *real* rate of return. Then the Fisher relationship becomes:

$$(1+y_n) = (1+\widetilde{y}_n^*)(1+\pi_n) \tag{7}$$

This requires an important assumption about investors' behaviour: that they ignore uncertainty (i.e., investors are *risk neutral*).

Notice that Z_n is unknown as of today: the increase in the price level can be calculated only when the the maturity date has arrived. Hence, the *real* return on *nominal* bonds is ***uncertain***, and the *nominal* return on *real* bonds is ***uncertain***. In summary:

	Spot yields	
	Nominal	Real
Nominal ZC bond	certain	*uncertain*
Real ZC bond	*uncertain*	certain

Assuming that investors ignore uncertainty about future inflation (i.e., about Z_n) then the (nominal or real) rates of return on the two bonds will be equal – otherwise, investors would perceive a profit opportunity.

Formally, it is often assumed that:

$$\text{nominal return on a nominal ZC bond} = \text{nominal return on a real ZC bond}$$

$$y_n = \widetilde{y}_n$$

Equivalently:

$$\text{real return on a nominal ZC bond} = \text{real return on a real ZC bond}$$

$$y_n^* = \widetilde{y}_n^*$$

Either of these equalities can be used to *infer* the 'market's expectations' about the average rate of inflation over the life of the bonds, i.e. about Z_n. To see this, use the definitions to write each yield in terms of Z_n:

$$(1+y_n)^n = \frac{m}{p_n} \quad \text{and} \quad (1+\widetilde{y}_n)^n = \frac{m \times Z_n}{\widetilde{p}_n} \tag{8}$$

Hence, assuming $y_n = \widetilde{y}_n$, it follows that $(1+y_n)^n = (1+\widetilde{y}_n)^n$, and:

$$(1+y_n)^n = \frac{m}{p_n} = (1+\widetilde{y}_n)^n = \frac{m \times Z_n}{\widetilde{p}_n} \tag{9}$$

Cancelling m and simplifying:

$$\widetilde{p}_n / p_n = Z_n \tag{10}$$

Thus, for example, if the price of the real ZC bond is, say, 50% higher than the nominal ZC bond, this reveals that the price level is expected to increase by a factor of 50% over the life of the two bonds.

All that remains to be done is to write $\widetilde{p}_n / p_n = Z_n$ in terms of spot yieds, using the definitions above:

$$\widetilde{p}_n / p_n = \frac{m/(1+\widetilde{y}_n^*)^n}{m/(1+y_n)^n} = \frac{(1+y_n)^n}{(1+\widetilde{y}_n^*)^n} = Z_n \tag{11}$$

Substituting for Z_n gives:

$$\frac{(1+y_n)^n}{(1+\widetilde{y}_n^*)^n} = (1+\pi_n)^n \quad \text{or} \quad (1+y_n)^n = (1+\widetilde{y}_n^*)^n (1+\pi_n)^n \tag{12}$$

Lastly, take the nth root of both sides to obtain:

$$(1+y_n) = (1+\widetilde{y}_n^*)(1+\pi_n) \tag{13}$$

which, as required, is just:

$$(1 + \text{nominal rate of return}) = (1 + \text{real rate of return}) \times (1 + \text{expected rate of inflation})$$

Example:

Two zero-coupon (ZC) bonds, labelled N and R will each mature 10 years from today. At maturity, bond N pays \$100. At maturity, bond R promises to pay the amount of cash needed to buy \$100 worth of goods and services at today's prices.

The market prices of N and R, respectively, are observed to be $p_{10} = \$50$ and $\widetilde{p}_{10} = 80$. Thus, *assuming that the expected spot yields on R and N are equal*, it can be inferred that the price level is expected to increase by 60% between today and 10 years from now: $Z_{10} = \widetilde{p}_{10}/p_{10} = 80/50 = 1.60$, i.e., an expected rate of inflation rate approximately equal to 4.81% per year: $(80/50)^{1/10} - 1 \approx 0.0481$.

In terms of rates of return:

$$y_{10} = \left(\frac{m}{p_{10}}\right)^{1/10} - 1 = \left(\frac{100}{50}\right)^{1/10} - 1 \approx 7.18\% \tag{14}$$

$$\widetilde{y}_{10} = \left(\frac{m}{\widetilde{p}_{10}}\right)^{1/10} - 1 = \left(\frac{100}{80}\right)^{1/10} - 1 \approx 2.26\% \tag{15}$$

That is (apart from a tiny rounding error):

$$\begin{aligned} (1+\text{nominal rate of return}) &= (1+\text{real rate of return}) \times (1+\text{expected rate of inflation}) \\ (1+0.0718) &= (1+0.026) \times (1+0.0481) \end{aligned} \tag{16}$$

The common approximation ignores the cross-product[3] $(0.026 \times 0.0481 = 0.00108706)$:

$$\begin{aligned} \text{nominal rate of return} &\approx \text{real rate of return} + \text{expected rate of inflation} \\ 0.0718 &\approx 0.026 + 0.0481 \end{aligned} \tag{17}$$

Warning!

The above calculations rely on the assumption that investors are *risk neutral*, so that the only market equilibrium is such that the *expected* spot yields (real or nominal) in the two bonds are equal – and all investors share the same beliefs.

If investors are assumed (as seems plausible) to be risk averse, it is no longer legitimate to infer the expected inflation rate using the prices of nominal and real bonds. Indeed, investors' expectations (well as their risk preferences – their aversion to risk) may differ from one another (heterogenous beliefs).

Suppose, in the above example, that an investor observes $\widetilde{p}_{10}/p_{10} = 80/50 = 1.60$ but expects the price level to increase by a factor 1.20. There is a (risky) profit opportunity. How could the investor exploit it?

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[3] With continuous compounding, the cross-product vanishes: $\ln(100/50)/10 \equiv \ln(100/80)/10 + \ln(80/50)/10$, or $6.93\% = 2.23\% + 4.70\%$ (apart from a tiny numerical rounding error).