

## EC372 Economics of Bond and Derivatives Markets

### Bounds on Option Prices

#### A Common Method of Proof

1. Make a proposition,  $\mathcal{A}$ .
2. Suppose the contrary, i.e.  $\text{not-}\mathcal{A}$ .
3. Show that  $\text{not-}\mathcal{A}$  permits an *arbitrage opportunity*.<sup>1</sup>
4. Hence,  $\text{not-}\mathcal{A}$  is incompatible with market equilibrium in frictionless markets.<sup>2</sup>
5. Therefore (frictionless) market equilibrium, implies that  $\mathcal{A}$  holds.

#### A Simple Example

1. Proposition:  $p \leq X/R$  (see chapter 18 for notation). In words: the premium for a European put option is never greater than the net present value of its exercise price.
2. Suppose the contrary:  $p > X/R$ . (Equivalently,  $Rp > X$ .)
3. Strategy: write one put option for  $p$  and make a loan of the proceeds (i.e. buy a risk-free bond or put the money in the bank).  
At date  $T$  (expiry date for the option), the loan is worth  $Rp$  (the deposit plus interest).  
Also at date  $T$ , either  $S_T \geq X$  or  $S_T < X$ .  
If  $S_T \geq X$  the option dies, unexercised, and the payoff equals  $Rp > X > 0$ . (From the hypothesis of step 2,  $Rp > X$ .)  
If  $S_T < X$ , the option is exercised, with a loss of  $X - S_T$  to the writer. The *maximum* loss equals  $X$  (if the underlying asset is worthless, i.e.  $S_T = 0$ ). By hypothesis (step 2),  $Rp > X$ . Hence,  $Rp - X > 0$ .
4. Hence,  $p > X/R$  permits an arbitrage opportunity: a zero initial outlay results in a payoff of at least  $Rp - X > 0$  in every possible outcome.
5. Therefore frictionless market equilibrium implies that  $p \leq X/R$ .

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<sup>1</sup>*Arbitrage opportunity*: an investment strategy that yields risk-free payoffs with zero initial capital outlay. Formally, an arbitrage opportunity is a portfolio that requires zero initial capital, and results in a non-negative payoff in every state with a positive payoff in at least one state.

<sup>2</sup>*Frictionless markets*: zero transaction costs; no institutional restrictions on trades. *Market equilibrium*: absence of arbitrage opportunities (i.e., the arbitrage principle holds).