EC501 Econometric Methods and Applications

Problem Set 4: Sketch Solutions

Large Sample Methods

1 (a) $\lim n^{-2} = 0$.

(b) $\lim(-1)^n$ does not exist – the sequence alternates between ± 1.

(c) $\lim(4 - 2n^{-1}) = 4$.

(d) $\lim e^{-n} = 0$.

(e) $\lim \ln 2^n = \lim(n \ln 2) \to \infty$ – the limit is not finite.

(f) $\lim n^{-2}(n+2)(n-3) = \lim \left(\frac{n^2 - n - 6}{n^2}\right) = \lim \left(1 - \frac{1}{n} - \frac{6}{n^2}\right) = 1$.

2. (a) The likelihood function is

$$\begin{aligned} L(\beta) &= \prod_{i=1}^{n} f(y_i) = \prod_{i=1}^{n} \beta \exp\{-\beta y_i\} = \beta^n \exp\left\{-\beta \sum_{i=1}^{n} y_i\right\} \\ \Rightarrow \quad \ln L(\beta) &= n \ln \beta - \beta \sum_{i=1}^{n} y_i \\ \Rightarrow \quad \frac{d \ln L(\beta)}{d\beta} &= \frac{n}{\beta} - \sum_{i=1}^{n} y_i. \end{aligned}$$

Equating the last expression to zero yields $\hat{\beta} = n / \sum_{i=1}^{n} y_i = 1/\bar{y}$ where $\bar{y} = \sum_{i=1}^{n} y_i / n$ is the sample mean.

(b) Plugging in the sample information gives $\hat{\beta} = 50/25 = 2$.

(c) We have $\hat{\beta} \overset{a}{\sim} N(\beta, I(\beta)^{-1})$ where $I(\beta) = -E[d^2 \ln L(\beta)/d\beta^2]$. From (a),

$$\frac{d^2 \ln L(\beta)}{d\beta^2} = -\frac{n}{\beta^2} \quad \Rightarrow \quad I(\beta) = \frac{n}{\beta^2}.$$

It follows that $\hat{\beta} \overset{a}{\to} N(\beta, \beta^2/n)$. (Note that it is in many ways better to write $\sqrt{n}(\hat{\beta} - \beta) \overset{d}{\to} N(0, \beta^2)$).

(d) The test can be performed using the Wald test statistic, which is given by the quadratic form

$$W = c(\hat{\beta})' \left[C(\hat{\beta}) I(\hat{\beta})^{-1} C(\hat{\beta})'\right]^{-1} c(\hat{\beta}) \overset{d}{\to} \chi^2_J.$$

In this case we have $c(\beta) = (\beta - 1)$, $C(\beta) = 1$, $I(\beta) = 50/1$, $I(\hat{\beta}) = 50/4$, and therefore

$$W = (2-1)' \left[1 \times \frac{4}{50} \times 1\right]^{-1} (2-1) = 12.5.$$

Alternativelly (and in some ways better), we can do

$$W^* = (2-1)' \left[1 \times \frac{1}{50} \times 1\right]^{-1} (2-1) = 50.0,$$

where $I(\beta)$ is evaluated under the null. The 5% critical value from a χ^2_1 distribution is approximately 3.841 and hence in both cases we reject the null hypothesis (that $\beta = 1$) against the alternative that $\beta \neq 1$.
The test can also be performed using a simple t-test:

$$t = \frac{\hat{\beta} - \beta}{se(\hat{\beta})} = \frac{2-1}{\sqrt{4/50}} = 3.5355,$$

or

$$t^* = \frac{\hat{\beta} - \beta}{se(\beta)} = \frac{2-1}{\sqrt{1/50}} = 7.0711.$$

The 5% critical value from a t_{49} distribution is approximately 2.009 and hence in both cases we reject the null hypothesis (that $\beta = 1$) against the alternative that $\beta \neq 1$. We can also use the critical value form the normal distribution, $z^{0.025} = 1.96$, but asymptotically it is indifferent.
Notice that $W = t^2$ and that $W^* = (t^*)^2$ and that the critical value of the χ^2_1 is also the square of the corresponding critical values of the normal distribution. Therefore, using W or t (or W^* or t^*) leads to the same conclusion.

3. (a) The likelihood function is

$$\begin{aligned} L(\theta) &= \prod_{i=1}^n f(x_i;\theta) = \prod_{i=1}^n (2\pi\theta)^{-1/2} \exp\{-x_i^2/2\theta\} \\ &= (2\pi\theta)^{-n/2} \exp\left\{-\sum_{i=1}^n x_i^2/2\theta\right\} \\ \Rightarrow \ln L(\theta) &= -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\theta - \frac{\sum_{i=1}^n x_i^2}{2\theta} \\ \Rightarrow \frac{d\ln L(\theta)}{d\theta} &= -\frac{n}{2\theta} + \frac{\sum_{i=1}^n x_i^2}{2\theta^2}. \end{aligned}$$

Equating the last expression to zero yields $\hat{\theta} = \sum_{i=1}^n x_i^2/n$ (which is the 'natural' estimator in this case).

(b) We have $\hat{\theta} \overset{a}{\sim} N(\theta, I(\theta)^{-1})$ where $I(\theta) = -E[d^2 \ln L(\theta)/d\theta^2]$. From (a),

$$\frac{d^2 \ln L(\theta)}{d\theta^2} = \frac{n}{2\theta^2} - \frac{\sum_{i=1}^n x_i^2}{\theta^3} \Rightarrow I(\theta) = -\frac{n}{2\theta^2} + \frac{\sum_{i=1}^n E(x_i^2)}{\theta^3} = -\frac{n}{2\theta^2} + \frac{n\theta}{\theta^3} = \frac{n}{2\theta^2}.$$

It follows that $\hat{\theta} \overset{a}{\sim} N(\theta, 2\theta^2/n)$. (Note that it is in many ways better to write $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, 2\theta^2)$.)

(c) Plugging in the numbers gives $\hat{\theta} = 110/100 = 1.1$.

(d) Unrestricted: $\ln L = -50\ln 2\pi - 50\ln 1.1 - (110/2.2) = -146.6594$;
Restricted: $\ln L_R = -50\ln 2\pi - 50\ln 1 - (110/2) = -146.8939$.
Hence $LR = -2(\ln L_R - \ln L) = 0.469$. The 5% critical value for the χ^2_1 is 3.84, hence LR is less than the critical value and we do not reject $H_0 : \theta = 1$.