UNIVERSITY OF ESSEX
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DEPARTMENT OF ECONOMICS
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EC501 Econometric Methods and Applications

Problem Set 6: Sketch Solutions

The Generalised Linear Regression Model and Heteroskedasticity

1. (a) OLS is unbiased, consistent, but not minimum variance.

   (b) This can be done using the Eicker-White heteroskedasticity consistent estimator of $var(b|X)$:

   $$\widehat{var}(b|X) = \left(\sum_{i=1}^{n} x_i x_i'\right)^{-1} \left(\sum_{i=1}^{n} e_i^2 x_i x_i'\right) \left(\sum_{i=1}^{n} x_i x_i'\right)^{-1},$$

   where $e_i = y_i - x_i'b$ are the OLS residuals.

   (c) An appropriate method is the modified Breusch-Pagan LM test, which is a test of $H_0 : \alpha = 0$ against $H_1 : \alpha \neq 0$ assuming $\sigma_i^2 = f(\alpha_0 + z_i'\alpha)$, where z_i is an observable $p \times 1$ vector of functions of the regressors. To perform the test, regress e_i^2 on $(1, z_i')$ and compute the R^2 of this auxiliary regression. Then

   $$nR^2 \sim \chi_p^2$$

   as $n \to \infty$ under H_0. Alternatively, use an F-statistic to test the null hypothesis that all slopes in the auxiliary regression are equal to zero.

   (d) Construct the diagonal matrix $\sigma^2\Omega = \text{diag}\{\sigma_i^2\}$ and compute the GLS estimator

   $$\hat{\beta} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y = \left(\sum_{i=1}^{n} x_i x_i'/\sigma_i^2\right)^{-1} \sum_{i=1}^{n} x_i y_i/\sigma_i^2.$$

   This is equivalent to estimating the transformed model

   $$\frac{y_i}{\sqrt{\sigma_i^2}} = \frac{x_i'}{\sqrt{\sigma_i^2}}\beta + \frac{\epsilon_i}{\sqrt{\sigma_i^2}}.$$

   (e) First, obtain an approximation to σ_i^2. For example, use OLS to estimate

   $$\ln\left(e_i^2\right) = \alpha_0 + \alpha_1 \hat{y}_i + a_2 \hat{y}_i^2 + u_i,$$

   and approximate σ_i^2 with $s_i^2 = \exp(\hat{\alpha}_0 + \alpha_1 \hat{y}_i + a_2 \hat{y}_i^2)$, where $\hat{y}_i = x_i'b$. Then estimate the transformed model

   $$\frac{y_i}{\sqrt{s_i^2}} = \frac{x_i'}{\sqrt{s_i^2}}\beta + \frac{\epsilon_i}{\sqrt{s_i^2}}.$$

   If the approximation of σ_i^2 that is used is good enough, the errors of the transformed model will be "less heteroskedastic" and therefore this estimator may be more efficient than OLS. However, because we do not have consistent estimates of σ_i^2, all the inference should still be based on an Eicker-White heteroskedasticity consistent estimator of the covariance matrix.

2. The OLS estimates are as follows:

Source	SS	df	MS
Model	25386.0168	1	25386.0168
Residual	16497.9238	98	168.346161
Total	41883.9407	99	423.070108

Number of obs =	100
F(1, 98) =	150.80
Prob > F =	0.0000
R-squared =	0.6061
Adj R-squared =	0.6021
Root MSE =	12.975

y	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
x	.582871	.0474654	12.28	0.000	.4886775	.6770644
_cons	3.990008	2.704989	1.48	0.143	-1.377955	9.35797

(a) The observed test statistic is F(1, 98) = 10.50, to which corresponds a p-value of 0.002. Hence, we reject the null hypothesis that x does not affect the variance of ϵ.

(b) The F-statistic for the auxiliary regression is again F(1, 98) = 10.50. Therefore, we can replicate the results of the Stata command.

(c) The new results are as follows:

y	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
x	.582871	.0430876	13.53	0.000	.4973651	.6683769
_cons	3.990008	1.739963	2.29	0.024	.5371075	7.442908

(d) How to obtain FGLS estimates depends on the assumptions we are willing to make about the pattern of heteroskedasticity. Using the method presented in the lecture, we get the following results:

ys	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
xs	.6327582	.0331191	19.11	0.000	.5670344	.698482
w	1.774227	.8795582	2.02	0.046	.0287726	3.519682

(e) From the OLS regression in part (c) we obtain the t-ratio

$$t = \frac{.582871 - 0.9}{.0430876} = -7.3601.$$

From the WLS regression in part (d) we obtain the t-ratio

$$t = \frac{.6327582 - 0.9}{.0331191} = -8.0691.$$

The 5% critical value is approximately 1.98. Therefore, in both cases, we reject the null that $\beta_2 = 0.9$ against the alternative that $\beta_2 \neq 0.9$ because $|t| > 1.98$.

3. (a) Expanding $S_*(\beta)$ we obtain

$$S_*(\beta) = y'\Omega^{-1}y - 2\beta'X'\Omega^{-1}y + \beta'X'\Omega^{-1}X\beta$$

because $y'\Omega^{-1}X\beta = \beta'X'\Omega^{-1}y$ (both are scalars). Differentiating with respect to β:

$$\frac{\partial S_*(\beta)}{\partial \beta} = -2X'\Omega^{-1}y + 2X'\Omega^{-1}X\beta = 2(X'\Omega^{-1}X\beta - X'\Omega^{-1}y).$$

Equating to zero and solving yields $\hat{\beta} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$ as required.

(b) To show the consistency of $\hat{\beta}$ it is enough to show that $\lim_{n\to\infty} E(\hat{\beta}) = \beta$ and $\lim_{n\to\infty} var(\hat{\beta}) = 0$. It is straightforward to show that $E(\hat{\beta}) = \beta$ and hence $\lim_{n\to\infty} E(\hat{\beta}) = \beta$. As for the variance, we know that $var(\hat{\beta}) = \sigma^2 E\left[(X'\Omega^{-1}X)^{-1}\right]$ and so

$$\lim_{n\to\infty} var(\hat{\beta}) = \lim_{n\to\infty}\left[\frac{\sigma^2}{n} E\left[\left(\frac{X'\Omega^{-1}X}{n}\right)^{-1}\right]\right] = 0 \times \mathcal{Q} = 0.$$

Hence, $\hat{\beta}$ is a consistent estimator of β.

(c) We can express $\sqrt{n}(\hat{\beta} - \beta)$ as

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{X'\Omega^{-1}X}{n}\right)^{-1}\left(\frac{X'\Omega^{-1}\epsilon}{\sqrt{n}}\right).$$

The first component converges to Q_*^{-1} and so

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \sigma^2 Q_*^{-1})$$

as $n \to \infty$, the limiting variance matrix being $\sigma^2 Q_*^{-1} Q_* Q_*^{-1} = \sigma^2 Q_*^{-1}$.

(d) The aim is to show that $\text{plim}[\sqrt{n}(\hat{\beta}_F - \beta) - \sqrt{n}(\hat{\beta} - \beta)] = 0$. Define the following matrices and vectors:

$$A_n = \left(\frac{X'\Omega^{-1}X}{n}\right)^{-1}, \quad b_n = \frac{X'\Omega^{-1}\epsilon}{\sqrt{n}},$$
$$\hat{A}_n = \left(\frac{X'\hat{\Omega}^{-1}X}{n}\right)^{-1}, \quad \hat{b}_n = \frac{X'\hat{\Omega}^{-1}\epsilon}{\sqrt{n}},$$

so that $\sqrt{n}(\hat{\beta} - \beta) = A_n b_n$ and $\sqrt{n}(\hat{\beta}_F - \beta) = \hat{A}_n \hat{b}_n$. We know that $\text{plim}(\hat{A}_n - A_n) = 0$ and $\text{plim}(\hat{b}_n - b_n) = 0$ so

$$\begin{aligned} \text{plim}[\sqrt{n}(\hat{\beta}_F - \beta) - \sqrt{n}(\hat{\beta} - \beta)] &= \text{plim}[\hat{A}_n\hat{b}_n - A_n b_n] \\ &= \text{plim}[(\hat{A}_n - A_n)\hat{b}_n] + \text{plim}[A_n(\hat{b}_n - b_n)] \\ &= 0 \cdot 0 + Q_*^{-1} \cdot 0 = 0 \end{aligned}$$

and hence the two estimators are asymptotically equivalent.