UNIVERSITY OF ESSEX — DEPARTMENT OF ECONOMICS
Session: 2011/12 — João Santos Silva

EC501 Econometric Methods and Applications

Problem Set 7: Sketch Solutions

Serial Correlation and Dynamic Models

1. (a) By repeated backward substitution:

$$\begin{aligned}
\epsilon_t &= \rho(\rho\epsilon_{t-2} + u_{t-1}) + u_t \\
&= \rho^2\epsilon_{t-2} + u_t + \rho u_{t-1} \\
&= \rho^2(\rho\epsilon_{t-3} + u_{t-2}) + u_t + \rho u_{t-1} \\
&= \rho^3\epsilon_{t-3} + u_t + \rho u_{t-1} + \rho^2 u_{t-2} \\
&\ \ \vdots \\
&= \rho^s\epsilon_{t-s} + u_t + \rho u_{t-1} + \ldots + \rho^{s-2}u_{t-s+2} + \rho^{s-1}u_{t-s+1} \\
\text{i.e. } \epsilon_t &= \rho^s\epsilon_{t-s} + \sum_{i=0}^{s-1}\rho^i u_{t-i} \quad (s > 0).
\end{aligned}$$

(b) Clearly $E(\epsilon_t) = 0$ while, applying the $var(\cdot)$ operator to $\epsilon_t = \rho\epsilon_{t-1} + u_t$ we obtain

$$\gamma_0 = var(\epsilon_t) = \rho^2 var(\epsilon_{t-1}) + 2cov(\rho\epsilon_{t-1}, u_t) + var(u_t).$$

Now $var(\epsilon_{t-1}) = \gamma_0$, $cov(\epsilon_{t-1}, u_t) = E(\epsilon_{t-1}u_t) = 0$ and $var(u_t) = \sigma_u^2$, so that $\gamma_0 = \rho^2\gamma_0 + \sigma_u^2$, and hence

$$\gamma_0 = \frac{\sigma_u^2}{1-\rho^2}.$$

Furthermore, $\gamma_s = cov(\epsilon_t, \epsilon_{t-s}) = E(\epsilon_t\epsilon_{t-s})$ can be obtained from the representation in part (a):

$$\begin{aligned}
E(\epsilon_t\epsilon_{t-s}) &= E\left[\left(\rho^s\epsilon_{t-s} + \sum_{i=0}^{s-1}\rho^i u_{t-i}\right)\epsilon_{t-s}\right] \\
&= \rho^s E(\epsilon_{t-s}^2) + \sum_{i=0}^{s-1}\rho^i E(u_{t-i}\epsilon_{t-s}) \\
&= \rho^s\gamma_0 = \frac{\rho^s\sigma_u^2}{1-\rho^2} = \gamma_s,
\end{aligned}$$

because $E(u_{t-i}\epsilon_{t-s}) = 0$ for $i = 0, \ldots, s-1$.

(c) It follows from part (b) that $\rho_s = \gamma_s/\gamma_0 = \rho^s$. Simple calculations yield:

ρ	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
0.1	1.00	0.10	0.01	0.001	0.0001	0.00001
0.9	1.00	0.90	0.81	0.729	0.6561	0.59049

Clearly, the process with $\rho = 0.9$ has the larger correlations and is, therefore, more correlated over time.

2. (a) Clearly, $E(\epsilon_t) = 0$ while

$$\begin{aligned}\gamma_0 = E(\epsilon_t^2) &= E(u_t^2 + 2\lambda u_t u_{t-1} + \lambda^2 u_{t-1}^2) \\ &= E(u_t^2) + 2\lambda E(u_t u_{t-1}) + \lambda^2 E(u_{t-1}^2) \\ &= \sigma_u^2(1+\lambda^2).\end{aligned}$$

Furthermore,

$$\begin{aligned}\gamma_1 = E(\epsilon_t \epsilon_{t-1}) &= E(u_t + \lambda u_{t-1})(u_{t-1} + \lambda u_{t-2}) \\ &= E(u_t u_{t-1}) + \lambda E(u_{t-1}^2) + \lambda E(u_t u_{t-2}) \\ &\quad + \lambda^2 E(u_{t-1} u_{t-2}) = \lambda \sigma_u^2, \\ \gamma_s = E(\epsilon_t \epsilon_{t-s}) &= E(u_t + \lambda u_{t-1})(u_{t-s} + \lambda u_{t-s-1}) \\ &= E(u_t u_{t-s}) + \lambda E(u_{t-1} u_{t-s}) + \lambda E(u_t u_{t-s-1}) \\ &\quad + \lambda^2 E(u_{t-1} u_{t-s-1}) = 0 \quad \text{for} \quad s > 1.\end{aligned}$$

(b) Obviously $\rho_0 = 1$ while from part (a), $\rho_s = 0$ for $s > 1$, leaving

$$\rho_1 = \frac{\gamma_s}{\gamma_0} = \frac{\lambda \sigma_u^2}{\sigma_u^2(1+\lambda^2)} = \frac{\lambda}{1+\lambda^2}.$$

Simple calculations give:

λ	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
0.5	1.00	0.40	0.00	0.00	0.00	0.00
2.0	1.00	0.40	0.00	0.00	0.00	0.00

This implies that two different parameters in the MA(1) can result in the same autocorrelation structure, which is a potential identification problem. Note, however, that $2 = 1/0.5$, and it is easy to show that any pair of coefficients, λ and $1/\lambda$, will result in identical autocorrelation functions in the MA(1). This generalises to higher-order MA processes.

3. (a) Given that $E(\epsilon_t|X) = 0$ it follows that

$$\begin{aligned}E(b|X) &= \beta + (X'X)^{-1}X'E(\epsilon|X) = \beta, \\ \text{and } E(b) &= E_X(\beta) = \beta,\end{aligned}$$

so that b is unbiased. However, it will not be the BLUE (best linear unbiased estimator) because we are dealing with a generalised linear regression model, for which we know that the GLS estimator is BLUE.

(b) The LM test for first-order serial correlation is most easily carried out by OLS estimation of the equation

$$e_t = x_t'\gamma + \rho e_{t-1} + u_t,$$

and computing the LM statistic as $LM = TR^2$. As $T \to \infty$, LM converges to a chi-squared random variable with one degree of freedom.

(c) We know that

$$\begin{aligned} y_t &= x_t'\beta + \epsilon_t \qquad t = 1, \ldots, T, \\ \epsilon_t &= \rho\epsilon_{t-1} + u_t \quad |\rho| < 1, \end{aligned}$$

and therefore $\epsilon_{t-1} = y_{t-1} - x_{t-1}'\beta$. Hence, we have that

$$\begin{aligned} y_t &= x_t'\beta + \rho\epsilon_{t-1} + u_t \\ y_t &= x_t'\beta + \rho\left(y_{t-1} - x_{t-1}'\beta\right) + u_t \end{aligned}$$

which can be written as

$$y_t = \gamma y_{t-1} + x_t'\alpha + x_{t-1}'\delta + u_t$$

where $\gamma = \rho$, $\alpha = \beta$, $\delta = -\rho\beta$ and $u_t = \epsilon_t - \rho\epsilon_{t-1}$. The parameter ρ can be estimated by the estimate of γ while the vector β can be estimated by the vector α. For this dynamic model to be equivalent to the original static model with AR(1) errors, the estimate of δ should also be restricted to ensure that $\delta = -\rho\beta$ (this is a non-linear restriction that could be incorporated using appropriate methods, e.g., maximum likelihood).

4. (a) The estimated equation, along with the serial correlation test statistic, are as follows:

```
      Source |       SS       df       MS              Number of obs =     200
-------------+------------------------------           F(  1,    198) = 1015.73
       Model |  1729.75671     1  1729.75671           Prob > F      =  0.0000
    Residual |  337.189254   198  1.70297603           R-squared     =  0.8369
-------------+------------------------------           Adj R-squared =  0.8360
       Total |  2066.94597   199  10.3866631           Root MSE      =   1.305

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          x2 |   1.011538   .0317391    31.87   0.000      .948948    1.074128
       _cons |   1.177759   .1923631     6.12   0.000     .7984155    1.557102
------------------------------------------------------------------------------

Breusch-Godfrey LM statistic:  98.28872  Chi-sq( 1)  P-value =  3.6e-23
```

The LM statistic is highly significant, rejecting the absence of first-order serial correlation in the residuals.

(b) The estimates and LM test are as follows:

```
      Source |       SS       df       MS              Number of obs =     199
-------------+------------------------------           F(  3,   195) =  729.49
       Model |  1887.62844     3  629.209481           Prob > F      =  0.0000
    Residual |  168.194683   195  .862536837           R-squared     =  0.9182
-------------+------------------------------           Adj R-squared =  0.9169
       Total |  2055.82313   198  10.3829451           Root MSE      =  .92873

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           y |
         L1. |   .7031674   .0506626    13.88   0.000     .6032505    .8030844
          x2 |
         --. |    1.01443   .0233042    43.53   0.000     .9684692    1.060391
         L1. |  -.7501685   .0560993   -13.37   0.000    -.8608077   -.6395292
       _cons |   .5394993   .1768839     3.05   0.003      .190648    .8883505
------------------------------------------------------------------------------

Breusch-Godfrey LM statistic:  .4563323  Chi-sq( 1)  P-value =  .4993
```

In the LM test we are unable to reject the null of no serial correlation in the residuals and so this equation appears (at least, based on this evidence) to be consistent with the original regression having a first-order serially correlated disturbance.

(c) The test statistic is below:

```
 ( 1)  _b[l.x2] = -_b[l.y]*_b[x2]

           F(1, 195) =       2.14
            Prob > F =       0.1448
```

The test is therefore unable to reject the hypothesis at the 5% level, thereby adding further support to the possibility that the original regression model has a first-order serially correlated disturbance.

(d) The answer is: yes! The test for serial correlation in the original model rejected the null of no serial correlation; the transformed (dynamic) model showed no evidence of a serially correlated disturbances; and the implied restriction on the coefficients was not rejected. All this suggests that the data is well described by a model of the form $y_t = \beta_1 + \beta_2 x_{t2} + \epsilon_t$ with $\epsilon_t = \rho\epsilon_{t-1} + u_t$.