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Session: 2011/12 — João Santos Silva

EC501 Econometric Methods and Applications

Problem Set 9: Sketch Solutions

Simultaneous Equations Models

1. (a) The order condition states that the number of excluded exogenous (and predetermined) variables must be at least as great as the number of included endogenous variables for the parameters of an equation to be identified (this is a necessary, but not a sufficient, condition). In the first equation there is one included endogenous variable as a regressor (y_2) and one excluded exogenous variable (x_2), so the equation is (just) identified. In the second equation there is one included endogenous variable as a regressor (y_1) and one excluded exogenous variable (x_1), so the equation is also (just) identified.

   (b) OLS would be biased and inconsistent because of the correlation that exists between the endogenous variables and the disturbances. For example, in the first equation, y_2 is included as a regressor, but we know from the reduced form of the model that y_2 depends on ϵ_1. Hence

$$\text{plim}\,\frac{1}{T}\sum_{t=1}^{T} y_{t2}\epsilon_{t1} \neq 0,$$

   thereby resulting in OLS being inconsistent.

   (c) Substituting the second equation into the first we obtain

$$\begin{aligned} y_1 &= \gamma_1\left(\gamma_2 y_1 + \beta_2 x_2 + \epsilon_2\right) + \beta_1 x_1 + \epsilon_1, \\ &= \gamma_1\gamma_2 y_1 + \beta_1 x_1 + \gamma_1\beta_2 x_2 + \epsilon_1 + \gamma_1\epsilon_2, \\ &= \frac{\beta_1}{1-\gamma_1\gamma_2}x_1 + \frac{\gamma_1\beta_2}{1-\gamma_1\gamma_2}x_2 + \frac{\epsilon_1+\gamma_1\epsilon_2}{1-\gamma_1\gamma_2}. \end{aligned}$$

   Now, replacing this result in the second equation we get

$$\begin{aligned} y_2 &= \gamma_2\left(\frac{\beta_1}{1-\gamma_1\gamma_2}x_1 + \frac{\gamma_1\beta_2}{1-\gamma_1\gamma_2}x_2 + \frac{\epsilon_1+\gamma_1\epsilon_2}{1-\gamma_1\gamma_2}\right) + \beta_2 x_2 + \epsilon_2, \\ &= \frac{\gamma_2\beta_1}{1-\gamma_1\gamma_2}x_1 + \frac{\gamma_2\gamma_1\beta_2}{1-\gamma_1\gamma_2}x_2 + \beta_2 x_2 + \gamma_2\frac{\epsilon_1+\gamma_1\epsilon_2}{1-\gamma_1\gamma_2} + \epsilon_2, \\ &= \frac{\gamma_2\beta_1}{1-\gamma_1\gamma_2}x_1 + \frac{\beta_2}{1-\gamma_1\gamma_2}x_2 + \frac{\epsilon_2+\gamma_2\epsilon_1}{1-\gamma_1\gamma_2}. \end{aligned}$$

   Therefore, the reduced form is

$$\begin{aligned} y_1 &= \frac{\beta_1}{1-\gamma_1\gamma_2}x_1 + \frac{\gamma_1\beta_2}{1-\gamma_1\gamma_2}x_2 + \frac{\epsilon_1+\gamma_1\epsilon_2}{1-\gamma_1\gamma_2}, \\ y_2 &= \frac{\gamma_2\beta_1}{1-\gamma_1\gamma_2}x_1 + \frac{\beta_2}{1-\gamma_1\gamma_2}x_2 + \frac{\epsilon_2+\gamma_2\epsilon_1}{1-\gamma_1\gamma_2}. \end{aligned}$$

(d) Step 1. Regress y_1 on x_1 *and* x_2 and obtain the fitted values, $\hat{y}_1$; regress y_2 on x_1 *and* x_2 and obtain the fitted values, $\hat{y}_2$.

Step 2. Regress y_1 on $\hat{y}_2$ and x_1 to obtain $\hat{\gamma}_{1,2SLS}$ and $\hat{\beta}_{1,2SLS}$; Regress y_2 on $\hat{y}_1$ and x_2 to obtain $\hat{\gamma}_{2,2SLS}$ and $\hat{\beta}_{2,2SLS}$.

(a) Using the order condition, we find the following results:

i. The first equation is (over) identified; the second equation is non-identified;

ii. The two equations are (just) identified;

iii. None of the equations are identified.

(b) The model is now

$$\begin{aligned} y_{t1} &= -\gamma_{21} y_{t2} - \beta_{31} x_{t3} + \epsilon_{t1}, \\ y_{t2} &= -\gamma_{12} y_{t1} - \beta_{12} x_{t1} + \epsilon_{t2}, \end{aligned}$$

Substituting the second equation into the first we obtain

$$\begin{aligned} y_{t1} &= -\gamma_{21} \left(-\gamma_{12} y_{t1} - \beta_{12} x_{t1} + \epsilon_{t2}\right) - \beta_{31} x_{t3} + \epsilon_{t1}, \\ &= \gamma_{21}\gamma_{12} y_{t1} + \gamma_{21}\beta_{12} x_{t1} - \beta_{31} x_{t3} + \epsilon_{t1} - \gamma_{21}\epsilon_{t2}, \\ &= \frac{\gamma_{21}\beta_{12}}{1-\gamma_{21}\gamma_{12}} x_{t1} - \frac{\beta_{31}}{1-\gamma_{21}\gamma_{12}} x_{t3} + \frac{\epsilon_{t1} - \gamma_{21}\epsilon_{t2}}{1-\gamma_{21}\gamma_{12}}. \end{aligned}$$

Now, replacing this result in the second equation we get

$$\begin{aligned} y_{t2} &= -\gamma_{12} \left(\frac{\gamma_{21}\beta_{12}}{1-\gamma_{21}\gamma_{12}} x_{t1} - \frac{\beta_{31}}{1-\gamma_{21}\gamma_{12}} x_{t3} + \frac{\epsilon_{t1} - \gamma_{21}\epsilon_{t2}}{1-\gamma_{21}\gamma_{12}} \right) - \beta_{12} x_{t1} + \epsilon_{t2}, \\ &= \frac{-\beta_{12}}{1-\gamma_{21}\gamma_{12}} x_{t1} + \frac{\gamma_{12}\beta_{31}}{1-\gamma_{21}\gamma_{12}} x_{t3} + \frac{\epsilon_{t2} - \gamma_{12}\epsilon_{t1}}{1-\gamma_{21}\gamma_{12}}. \end{aligned}$$

Therefore, the reduced form is

$$\begin{aligned} y_{t1} &= \frac{\gamma_{21}\beta_{12}}{1-\gamma_{21}\gamma_{12}} x_{t1} - \frac{\beta_{31}}{1-\gamma_{21}\gamma_{12}} x_{t3} + \frac{\epsilon_{t1} - \gamma_{21}\epsilon_{t2}}{1-\gamma_{21}\gamma_{12}}, \\ y_{t2} &= \frac{-\beta_{12}}{1-\gamma_{21}\gamma_{12}} x_{t1} + \frac{\gamma_{12}\beta_{31}}{1-\gamma_{21}\gamma_{12}} x_{t3} + \frac{\epsilon_{t2} - \gamma_{12}\epsilon_{t1}}{1-\gamma_{21}\gamma_{12}}. \end{aligned}$$

(a) This model has three types of variables:
Endogenous variables: C_t, I_t, W_t^P, X_t, P_t and K_t;
Exogenous variables: A_t, W_t^G, T_t and G_t;
Predetermined variables: X_{t-1}, P_{t-1} and K_{t-1}.

(b) The regression output is

Source	SS	df	MS
Model	294.248018	7	42.0354311
Residual	61.9500944	13	4.76539188
Total	356.198112	20	17.8099056

Number of obs	=	21
F(7, 13)	=	8.82
Prob > F	=	0.0004
R-squared	=	0.8261
Adj R-squared	=	0.7324
Root MSE	=	2.183

p	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
a	.3194049	.7781286	0.41	0.688	-1.36164	2.000449
wg	-.0796076	2.533823	-0.03	0.975	-5.5536	5.394385
g	.4390162	.3911427	1.12	0.282	-.4059962	1.284029
t	-.9230977	.4337595	-2.13	0.053	-1.860178	.0139827
p1	.8025008	.5188558	1.55	0.146	-.318419	1.923421
x1	.0220002	.2821641	0.08	0.939	-.5875783	.6315787
k1	-.2161035	.1191134	-1.81	0.093	-.4734323	.0412253
_cons	50.38438	31.63026	1.59	0.135	-17.94863	118.7174

and the results of the test are `F(5, 13) = 1.93`, to which corresponds a p-value of 0.1566. Becasue this is a test for $\Pi_j^* = 0$, the result suggests that there may be an identification problem.

(c) These are 2SLS estimates of the structural parameters in the equation for I_t. The standard errors are incorrect and they are useless. The results are as follows

Source	SS	df	MS
Model	211.188724	3	70.3962413
Residual	41.1379346	17	2.41987851
Total	252.326659	20	12.6163329

Number of obs	=	21
F(3, 17)	=	29.09
Prob > F	=	0.0000
R-squared	=	0.8370
Adj R-squared	=	0.8082
Root MSE	=	1.5556

i	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
phat	.1502219	.229128	0.66	0.521	-.3331959	.6336396
p1	.6159435	.215314	2.86	0.011	.1616707	1.070216
k1	-.1577876	.0477837	-3.30	0.004	-.2586024	-.0569729
_cons	20.27821	9.976633	2.03	0.058	-.7706448	41.32707

(d) The estimated parameters are the same but, as expected, the standard errors are smaller. The results are as follows

```
      Source |       SS       df       MS              Number of obs =      21
-------------+------------------------------           F(  3,    17) =   41.20
       Model |  223.279812     3  74.4266041           Prob > F      =  0.0000
    Residual |  29.0468463    17  1.70863802           R-squared     =  0.8849
-------------+------------------------------           Adj R-squared =  0.8646
       Total |  252.326659    20  12.6163329           Root MSE      =  1.3071

------------------------------------------------------------------------------
           i |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           p |   .1502219   .1925335     0.78   0.446    -.2559883    .5564322
          p1 |   .6159434   .1809258     3.40   0.003     .2342234    .9976635
          k1 |  -.1577876   .0401521    -3.93   0.001    -.2425011   -.0730742
       _cons |   20.27821   8.383247     2.42   0.027     2.591104    37.96531
------------------------------------------------------------------------------
Instrumented:  p
Instruments:   p1 k1 a x1 t g wg
```

(e) The OLS results are substantially different from those of the 2SLS, suggesting an important endogeneity bias. The results are as follows

```
      Source |       SS       df       MS              Number of obs =      21
-------------+------------------------------           F(  3,    17) =   76.88
       Model |   235.00396     3  78.3346533           Prob > F      =  0.0000
    Residual |  17.3226985    17  1.01898226           R-squared     =  0.9313
-------------+------------------------------           Adj R-squared =  0.9192
       Total |  252.326659    20  12.6163329           Root MSE      =  1.0094

------------------------------------------------------------------------------
           i |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           p |   .4796356   .0971146     4.94   0.000     .2747418    .6845294
          p1 |   .3330387   .1008592     3.30   0.004     .1202444     .545833
          k1 |  -.1117947   .0267276    -4.18   0.001    -.1681849   -.0554045
       _cons |   10.12579   5.465546     1.85   0.081    -1.405502    21.65709
------------------------------------------------------------------------------
```