Ec951, 2009. Instructor: David Reinstein
Hidden information: Practice Questions

1 Hidden information in the principal-agent setting.

Suppose a principal (P) delegates production to an agent (A), and pays the agent $t(q)$ to produce q units. Production incurs no fixed cost and a constant marginal cost. The marginal cost can have two types, $\underline{\theta}$ and $\bar{\theta}$, each of which occur with some nonzero probability. Assume $0 < \underline{\theta} < \bar{\theta} < 1$. Assume A's reservation utility is zero.

Assume:

$$U_P(q) = S(q) - t(q) = q^{1/2} - t(q)$$
$$U_A(q) = t(q) - c(q) = t(q) - \theta q \text{ where } \theta \in \{\underline{\theta}, \bar{\theta}\}$$

(a) First assume that P knows A's marginal cost before writing a contract. **Describe the 'first-best' contract and outcome.**

Suggested Answer: As P observes marginal cost and has all the bargaining power, he can induce the output that maximizes the total surplus, and capture this entire surplus. The surplus is maximized when marginal cost equals marginal utility – the first order condition of this concave problem. Hence $\frac{\partial U}{\partial q} = \frac{1}{2}q^{-1/2} = \theta$ $\frac{1}{2}q^{-1/2} = \theta \Longrightarrow q^* = \frac{1}{4\theta^2}$ where $\theta \in \{\underline{\theta}, \bar{\theta}\}$ Since costs are observed, the principal can set a different contract if he observes $\underline{\theta}$ or $\bar{\theta}$. These contracts need only meet the participation constraints – the incentive compatibility constraints can be trivially satisfied through a 'forcing contract'. $PC(\underline{\theta}) : t_{lc} - \underline{\theta} q_{lc} \geq 0$ $PC(\bar{\theta}) : t_{hc} - \bar{\theta} q_{hc} \geq 0$. It is trivial to see that these constraints will both bind. Hence, the contract: $t_{hc} = t(q = \frac{1}{4\underline{\theta}^2} | \theta = \underline{\theta}) = \frac{\underline{\theta}}{4\underline{\theta}^2} = \frac{1}{4\underline{\theta}}$, $t(q \neq \frac{1}{4\underline{\theta}^2} | \theta = \underline{\theta}) = -\infty$ and $t_{lc} = t(q = \frac{1}{4\bar{\theta}^2} | \theta = \bar{\theta}) = \frac{\bar{\theta}}{4\bar{\theta}^2} = \frac{1}{4\bar{\theta}}$, $t(q \neq \frac{1}{4\bar{\theta}^2} | \theta = \bar{\theta}) = -\infty$ will be optimal.

(b) Now assume the agent knows her marginal cost of production, but the principal does not. **Assuming the principal wants to implement an "interior, separating equilibrium," describe the principal's constrained optimization problem.**

Suggested Answer: To do this the principal will have to satisfy participation constraints and incentive compatibility constraints for both types (although only one of each will bind).

$$PC(\bar{\theta}) : t_{hc} - \bar{\theta} q_{hc} \geq 0$$
$$PC(\underline{\theta}) : t_{lc} - \underline{\theta} q_{lc} \geq 0$$
$$IC(\bar{\theta}) : t_{hc} - \bar{\theta} q_{hc} \geq t_{lc} - \bar{\theta} q_{lc}$$
$$IC(\underline{\theta}) : t_{lc} - \underline{\theta} q_{lc} \geq t_{hc} - \underline{\theta} q_{hc}$$

P's problem:

$$\max_{t_{hc},t_{lc},q_{hc},q_{lc}} E(q^{1/2} - t(q))$$
$$= \nu \times (q_{hc}^{1/2} - t_{hc}) + (1-\nu) \times (q_{lc} - t_{lc})$$
$$\text{s.t. } PC(\bar{\theta}), PC(\underline{\theta}),\ IC(\bar{\theta}), IC(\underline{\theta})$$

Where $prob(\theta = \bar{\theta}) = \nu$.

(c) Which of the four constraints described in (b) will bind, and why? Explain intuitively. For one of these constraints, show a proof that it will or will not bind.

Suggested Answer:

[*Note: less formal proofs will suffice on an exam*]

The participation constraint for the high-cost type will bind, while the incentive compatibility constraint for the low-cost type will bind. The low-cost type will get an information rent to keep her from pretending she is the high-cost type. The high-cost type will get no rent, she will have no incentive to pretend to be low-cost and thus gets no rent, and the PC can be met with equality.

In general, we can show that if we give just enough so that the high-cost type wants to participate (zero), if the low-cost type is also given zero she'd prefer to pretend to be the high-cost type, produce this lower output, and save on costs, earning a positive surplus. The converse is not true – at the low-cost type's threshold of participation the high-cost type does not benefit from pretending to be the low-cost type and producing a higher output – the difference in marginal costs is unfavorable for her. Hence the low-cost type must get an information rent, but the high-cost type need not. Since the information rent is a function of the 'difference' in payoffs to the two agents, we will thus want to bring the high-cost's surplus to zero.

$PC(\underline{\theta})$ will not bind.

"Proof": The low-cost type must receive a positive surplus – an "information rent" (in an interior, separating contract)

The low-cost type can always mimic the high-cost type and earn a surplus ($\Delta\theta q_{hc}$ below)– we need to give her at least this in order to 'be herself.'

Suppose q_{hc},$q_{lc}$$> 0 \Longrightarrow u(\bar{\theta}, q_{hc}) = t_{hc} - \bar{\theta} q_{hc} \geq 0$ needed to satisfy the high-type's PC $\Longrightarrow$If low-type accepts this ('lies'), gets

$$\begin{aligned} u(\underline{\theta}, q_{hc}) &= t_{hc} - \underline{\theta} q_{hc} \\ &= t_{hc} - \bar{\theta} q_{hc} + \Delta\theta q_{hc} \\ &= u(\bar{\theta}, q_{hc}) + \Delta\theta q_{hc} \\ &> 0 \end{aligned}$$

Where $\Delta\theta \equiv \bar{\theta} - \underline{\theta}$

$PC(\bar{\theta})$ is a binding constraint

Proof (by contradiction): Suppose that $\bar{U} = t_{hc} - \bar{\theta} q_{hc} > 0$. Then P could lower t_{hc} by "ε", increase P 's expected surplus, and the high-cost type would still participate. I.e., $PC(\bar{\theta})$ would still hold since it was previously slack. Since $PC(\underline{\theta})$ doesn't bind (information rent) P could also lower t_{lc} by this same amount "ε", preserving $IC(\underline{\theta})$ and $IC(\bar{\theta})$.

$IC(\underline{\theta})$ is a binding constraint "Proof" (by contradiction): If not, i.e., $t_{lc} - \underline{\theta} q_{lc} > t_{hc} - \bar{\theta} q_{hc}$, P could lower t_{lc} and still satisfy the relevant constraints: $IC(\underline{\theta})$ would continue to hold (if wasn't binding). $IC(\bar{\theta})$ is *relaxed* (*less* incentive for high-cost type to 'fake it'). $PC(\underline{\theta})$ would continue to hold because it didn't bind before– remember we showed an 'information rent.' $PC(\bar{\theta})$ unaffected.

$IC(\bar{\theta})$ is not binding. Intuition: since $IC(\underline{\theta})$ is binding , the difference in surplus for the low cost type pretending to be high-cost is zero. Even for the low-cost guy, producing high output and getting t_{hc} is only weakly better (i.e., neutral) then producing low output. If this is the case, producing high output and getting t_{hc} is surely not worth doing for the high-cost type, as it would be more expensive for him to do so.

Proof:

High-cost type gets $t_{hc} - \bar{\theta} q_{hc}$ for being herself versus $t_{lc} - \bar{\theta} q_{lc}$ if pretends to be low-cost type.

The relative payoff for the high-cost type taking the q_{hc} contract is:
$(t_{hc} - \bar{\theta} q_{hc}) - (t_{lc} - \bar{\theta} q_{lc})$.

The relative payoff for the low-cost type taking the q_{hc} contract is:
$(t_{hc} - \underline{\theta} q_{hc}) - (t_{lc} - \underline{\theta} q_{lc})$
$= 0$ since $IC(\underline{\theta})$ is binding

The low-cost type is indifferent. For the high-cost type to want to produce q_{hc} she needs the relative payoff difference to be at least as great as for the low-cost type.

I.e., we need $[(t_{hc} - \bar{\theta} q_{hc}) - (t_{lc} - \bar{\theta} q_{lc})] - [(t_{hc} - \underline{\theta} q_{hc}) - (t_{lc} - \underline{\theta} q_{lc})] \geq 0$

The transfers will cancel out, since these are the same for both types for a given output.

Hence, $[\bar{\theta} q_{lc} - \bar{\theta} q_{hc}] - [\underline{\theta} q_{lc} - \underline{\theta} q_{hc}] \geq 0$

Rearranging: $(\bar{\theta} - \underline{\theta}) q_{lc} - (\bar{\theta} - \underline{\theta}) q_{hc} = \Delta\theta \times (q_{lc} - q_{hc}) > 0$ since $q_{lc} > q_{hc}$.

So, when the low-cost type is indifferent between producing low and high output, the high-cost type strictly prefers producing high output.

(d) Now assume the following parameter values:

$\underline{\theta} = \frac{1}{4}$
$\bar{\theta} = \frac{1}{2}$
Each type of agent occurs with $\frac{1}{2}$ probability, i.e., $prob(\theta = \underline{\theta}) = \frac{1}{2} = prob(\theta = \bar{\theta})$

Solve for P's optimal 'second-best' (separating, interior) contract (transfers and outputs). Contrast this outcome to that of part (a), assuming these numerical parameter values held in part (a). Discuss.

Note: You can plug in these parameters to get the corresponding numerical values for part a.

Suggested Answer: As proved above, $PC(\bar{\theta})$ and $IC(\underline{\theta})$ are binding, thus must hold with equality. Hence $t_{hc} = \bar{\theta} q_{hc} = \frac{q_{hc}}{2}$ and $t_{lc} = t_{hc} - \underline{\theta} q_{hc} + \underline{\theta} q_{lc} = \underline{\theta} q_{lc} + (\bar{\theta} - \underline{\theta}) q_{hc} = \frac{q_{lc}}{4} + \frac{q_{hc}}{4} = \frac{q_{hc}+q_{lc}}{4}$. All outputs other than q_{lc} or q_{hc} get a zero or negative payoff, let's say. We still need to solve for the optimal q_{lc} and q_{hc}, i.e., q_{lc} and q_{hc}^{SB}. Substituting in these constraints, P solves:

$$\max_{q_{hc}, q_{lc}} E(q^{1/2} - t(q))$$
$$= \frac{1}{2}(q_{hc}^{1/2} - \frac{q_{hc}}{2} + q_{lc}^{1/2} - \frac{q_{hc} + q_{lc}}{4})$$

Taking the first order conditions of this concave problem:

$$2\frac{\partial EU_p}{\partial q_{lc}} = \frac{1}{2} q_{lc}^{-1/2} - \frac{1}{4} = 0$$
$$\Longrightarrow q_{lc}^{SB} = (\frac{1}{2})^{-2} = 4$$
$$\text{remember and note } q_{lc}^{*} = \frac{1}{4\underline{\theta}^2} = \frac{1}{4 * \frac{1}{16}} = 4$$

$$2\frac{\partial EU}{\partial q_{hc}} = \frac{1}{2} q_{hc}^{-1/2} - \frac{1}{2} - \frac{1}{4} = 0$$
$$q_{hc}^{-1/2} = \frac{6}{4}$$
$$\Longrightarrow q_{hc}^{SB} = (\frac{6}{4})^{-2} = \frac{4}{9}$$
$$\text{remember and note } \ q_{hc}^{*} = \frac{1}{4\bar{\theta}^2} = \frac{1}{4 * \frac{1}{2}} = \frac{1}{2} > \frac{4}{9}$$

The optimal second-best contract thus offers the low cost type the first best quantity while it distorts the quantity of the high cost agent downward to minimize informational rents. As seen in the FOC above, increasing high-cost type's quantity brings two costs – the direct cost $\bar{\theta}$ and the required increase in payment to the low-cost type $(\bar{\theta} - \underline{\theta})$.

Solving for transfers: $t_{hc}^{SB} = \bar{\theta} q_{hc}^{SB} = \frac{q_{hc}}{2} = \frac{2}{9}$ $t_{lc}^{SB} = \underline{\theta} q_{lc} + (\bar{\theta} - \underline{\theta}) q_{hc} = 4\frac{1}{4} + \frac{1}{4}\frac{4}{9} = \frac{10}{9}$.

Solving for principal's expected utility: $EU_p^{SB} = \frac{1}{2}(4^{\frac{1}{2}} - \frac{10}{9}) + \frac{1}{2}(\frac{4}{9}^{\frac{1}{2}} - \frac{2}{9}) = \frac{2}{3}$

(e) Define the 'shut-down one type' strategy and the 'pooling' strategy. Show that, for the case above, P prefers to implement an interior, separating equilibrium.

Suggested Answer: Shut-down: In the 'shut down' strategy the principal writes a contract that only the low-cost agent will sign. Here, the principal can set the quantity that maximizes the surplus from the low-cost agent and capture all of this surplus. The principal does not need to worry about the high-cost type nor about the incentive compatibility – there is only one level output that is rewarded.

Pooling: In the 'pooling' strategy the principal has both types producing the same amount, offering a reward for this amount and this amount only. Thus, only participation constraints matter, in particular the high-cost type's participation constraint, since her costs are higher to produce the specified amount.

In the case above, shutting down the high-cost type: $PC(\underline{\theta}) : t_{lc} - \underline{\theta}q_{lc} \geq 0$ is the only constraint, and it is easy to see it is binding.

Hence$t_{lc}^{shut} = \underline{\theta}q_{lc} = \frac{q_{lc}}{4}$ The optimal output from the low-cost type is $q^* = \frac{1}{4\underline{\theta}^2} = 4$

Hence the transfer $t_{lc}^{shut} = \underline{\theta}q_{lc} = \frac{q_{lc}}{4} = 1$

Hence, the principal's expected utility is $EU_p^{shut} = \frac{1}{2}(4^{\frac{1}{2}} - 1) = \frac{1}{2}$, noting there is only a $\frac{1}{2}$ probability of encountering a low-cost type.

In the case above, pooling: $PC(\bar{\theta}) : t^{pool} - \bar{\theta}q^{pool} = t^{pool} - \frac{q^{pool}}{2} \geq 0$ is the only binding constraint given the structure of such a contract.

Hence, $t^{pool} = \frac{q^{pool}}{2}$ The principal is maximizing subject to the high-cost type's participation. She captures all of the surplus from this production.

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s the q^{pool} that will be optimal is the one that is technically efficient for the high-cost type.

Hence, $q^{pool} = \frac{1}{4\bar{\theta}^2} = \frac{1}{4\frac{1}{2}^2} = 1$

Hence $t^{pool} = \frac{q^{pool}}{2} = \frac{1}{2}$

Hence, the principal's expected utility is $EU_p^{pool} = (1^{\frac{1}{2}} - \frac{1}{2}) = \frac{1}{2}$ noting the payoff is the same for either type of agent. Comparing these three possibilities:

$EU_p^{SB} = \frac{2}{3}$, $EU_p^{shut} = \frac{1}{2}$,$EU_p^{pool} = \frac{1}{2}$.

We see that in this case the interior separating contract is preferred.

Consider: What if there is a cost to writing a contract clause? What if there is a fixed cost for hiring each employee?

Consider (tricky): How much would the principal have to offer an agent to get her to certainly reveal her type? How would this work? What if the agent could "prove" high costs?

Consider (tricky): How much would a high-cost agent be willing to invest in human capital (in the above example) in order to become a low-cost type? Would it matter whether this investment was observable to the principal? Suppose a high-cost agent could make an investment that would give her a 50% probability of becoming a low-cost type. Suppose this investment was observable to the principal but only the agent would learn whether the investment was "successful." How much would the agent be willing to pay to make such an investment?

2 Hidden information in the principal-agent setting; some numerical values.

Answer *all* parts (a), (b) and (c) of this question.

Consider a standard hidden-information agency problem in which a principal hires an agent. Define a contract as a pairing of a transfer (t) and an output level (q). The principal offers a menu of one or more contracts. The agent knows his type, $\theta \in \{1, 2\}$, at the time she is hired. The probability that $\theta = 2$ is $\frac{2}{3}$. This is her private information. For a given contract (t, q) the agent's utility is $t - \theta q^2$ and the principal's utility is $q - t$. The output level is observable.

(a) Suppose the principal can observe the type of the agent before offering her a contract. Solve for the contract that the principal will write and find the resulting payoffs.

Suggested answer: In the full-information case, the principal, knowing θ, maximizes $q - t$ over q and t subject to the constraint that $t - \theta q^2 \geq 0$

(i.e., subject to the constraint that the agent participate).

Clearly, the constraint binds — the principal wishes to make t as small as possible — so the problem is equivalent to the unconstrained problem

$$\max_q \, (q - \theta q^2)$$

The solution to which is $q^{FB}(\theta) = \frac{1}{2\theta}$

Hence, $t^{FB}(\theta) = \frac{1}{4\theta}$.

I.e.,

$$\begin{aligned} q^{FB}(\underline{\theta}) &= \frac{1}{2\underline{\theta}} = \frac{1}{2} \\ t^{FB}(\underline{\theta}) &= \frac{1}{4\underline{\theta}} = \frac{1}{4} \end{aligned}$$

and

$$\begin{aligned} q^{FB}(\bar{\theta}) &= \frac{1}{2\bar{\theta}} = \frac{1}{4} \\ t^{FB}(\bar{\theta}) &= \frac{1}{4\bar{\theta}} = \frac{1}{8}. \end{aligned}$$

Payoffs yield zero surplus to each agent, and a (principal's) profit of:

$$\begin{aligned} U_{pr}^{FB} &= \frac{1}{3}\left(q_{lc}-t_{lc}\right)+\frac{2}{3}\left(q-t_{hc}\right) \\ &= \frac{1}{3}(\frac{1}{2}-\frac{1}{4})+\frac{2}{3}(\frac{1}{4}-\frac{1}{8})=\frac{1}{6} \end{aligned}$$

Note: Another way of stating this is that these contracts need only meet the participation constraints – the incentive compatibility constraints can be trivially satisfied through a 'forcing contract'. $PC(\underline{\theta}) : t_{lc} - \underline{\theta} \underline{q}^2 \geq 0$; $PC(\bar{\theta}) : t_{hc} - \bar{\theta}\bar{q}^2 \geq 0$. It is trivial to see that these constraints will both bind.

Hence, the contract: $t_{lc} = t(q = \frac{1}{4\underline{\theta}} | \theta = \underline{\theta}) = \frac{1}{8}$, $t(q \neq \frac{1}{4\underline{\theta}} | \theta = \underline{\theta}) = -\infty$ and $t_{hc} = t(q = \frac{1}{4\bar{\theta}} | \theta = \bar{\theta}) = \frac{1}{4}$, $t(q \neq \frac{1}{4\bar{\theta}} | \theta = \bar{\theta}) = -\infty$ will be optimal.

(b)Now assume the principal does not observe the agent's type, but knows the probability of the agent being of each type. Find the optimal menu of contracts for the principal to offer. Find the resulting payoffs. Remember to check shut-down and pooling contracts. Show how you obtained your answer. Note: you do not need to give a proof of which constraints bind and which are slack, you can merely cite a general rule.

Suggested answer:

Separating case

We know that to induce a separating contract with participation we need to only worry about the low-cost type's IC constraint and the high-cost type's PC constraint; these are the binding constraints, and the others are 'slack.' Hence, we require:

$$t_{hc} - 2{q_{hc}}^2 \geq 0 \qquad \text{(PC-high)}$$

and

$$t_{lc} - q_{lc}^2 \geq t_{hc} - q_{hc}^2 \qquad \text{(IC-low)}$$

Since these are binding, they hold strictly, implying:

$$t_{hc} = 2{q_{hc}}^2$$

and

$$\begin{aligned} t_{lc} &= t_{hc} - q_{hc}^2 + q_{lc}^2 \\ &= 2q_{hc}^2 - q_{hc}^2 + q_{lc}^2 \qquad (1) \\ &= q_{hc}^2 + q_{lc}^2 \qquad (2) \end{aligned}$$

Substituting this into the principal's optimization problem we get:

$$\begin{aligned} U_{pr}^{SB} &= \max_{q,q_{lc}} E_{\theta}(q(\theta) - t(\theta)) \\ &= \max_{q,q_{lc}} \frac{1}{3}(q_{lc} - q_{hc}^2 - q_{lc}^2) + \frac{2}{3}\left(q_{hc} - 2q^2\right) \end{aligned}$$

Taking the FOC's of this concave problem:

$$\begin{aligned} \frac{dU_{pr}}{dq_{lc}} &= \frac{1}{3} - \frac{2}{3} q_{lc} = 0 \\ &\Rightarrow q_{lc}^{SB} = \frac{1}{2} = q^{FB}(\underline{\theta}) \end{aligned}$$

$$\begin{aligned} \frac{dU_{pr}}{dq_{hc}} &= -\frac{2}{3} q_{hc} + \frac{2}{3} - \frac{8}{3} q_{hc} = 0 \\ &\Rightarrow q_{hc}^{SB} = \frac{1}{5} \end{aligned}$$

Expected profits can be solved for:

$$\begin{aligned} t_{hc} &= q_{hc}^2 \times \bar{\theta} = \frac{1}{5}^2 \times 2 = \frac{2}{25} \\ t_{lc} &= t_{hc} - q_{hc}^2 + q_{lc}^2 = \frac{2}{25} - \frac{1}{25} + \frac{1}{2}^2 = \frac{1}{25} + \frac{1}{4} = 0.29 \\ U_{pr}^{SB} &= \frac{1}{3}(q_{lc} - t_{lc}) + \frac{2}{3}(q_{hc} - t_{hc}) \\ &= \frac{1}{3}\left(\frac{1}{2} - \frac{29}{100}\right) + \frac{2}{3}(\frac{1}{5} - \frac{2}{25}) = \frac{3}{20} \end{aligned}$$

Which is lower than in the full-information case.

(c) In the context of a standard hidden-information agency problem, what is an 'information rent' and why does it occur? If the principal implements a contract in which both types produce the same amount, will either type get an information rent? Explain why or why not.

Suggested answer: If a separating contract is optimal , the principal is writing a contract in which each type of agent (self-selects to) produces a different amount and is paid a different amount. The low-cost agent is paid an information rent to get her to produce more, i.e., to keep her from accepting the contract meant for the high-cost type. We could show that the PC_high is binding, and this will imply that if the low-cost type were to accept the high-cost type's contract she could do it at lower cost and get a positive profit. Thus, to accept the contract meant for the low-cost type, she must also get a positive profit, the 'information rent.' It is called an 'information rent' because it can be seen as a payment required to get the low-cost type to reveal her information about cost – if no such reward were given, she would rather remain 'incognito.'