EC951 Lecture 2: Moral Hazard part II

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Lecture notes 2: Multiple outcomes, MLRP, Signals/Informativeness

Reading: L&M 4.5, 4.6

Optional/useful: L&M 4.7

Alt: Milgrom and Roberts ch 7.

Exercises (pertaining to these notes):

L&M "moral hazard practice questions.pdf" – 1.3, 1.4b and 1.4c M&R "food for thought chapter 7, questions 1 and 2

2004 final, question 1: `Agents' compensation should not depend upon observable luck." Explain clearly why this is the case...`

Important reminder:

You **must** read about limited liability on your own (for now).

This is covered in L&M 4.3 for the 2x2x2 type case.

And in L&M 4.5.1 for the multiple outcome case.

For an exercise on this, see "moral hazard practice questions.pdf"; 1.1 (especially part b), 1.3a

Discussion question:

Why, when the likelihood ratios are different for each level of output, is the structure of the optimal contract 'bang-bang' in the LL/risk neutral case (only one output rewarded), while, in the risk-aversion case several levels of output may be rewarded?

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     - ⋆ Note L&M's 'number-line' diagram's (e.g., fig. 4.2, 4.3, 4.4 b) for this latter point.

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Outputs occur with probabilities $[\pi_{10}, \pi_{20}, \pi_{30}, ..., \pi_{n0}]$ if $e = 0$ and

$[\pi_{11}, \pi_{21}, \pi_{31}, ..., \pi_{n1}]$ if $e = 1$.

The maximization problem for the principal is given by

$$\max_{t_1, t_2, ..t_n} \sum_{i=1}^{n} \pi_{i1}[S_i - t_i] \tag{1}$$

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Recognizing that $t_i = h(u_i) = u^{-1}(u_i)$, we can express P's problem in terms of A's utility from the wage.

... if P wants to minimize the expected wage he also wants to minimize (the expectation of a transformation of) A's surplus. This shows the problem is concave subject to a linear constraint.):

$$\max_{u_1,u_2,..u_n} \sum_{i=1}^{n} \pi_{i1}[S_i - h(u_i)] \tag{2}$$

Going back to the previous notation, a contract is an n-tuple $\{t_1, ..., t_n\}$ or equivalently $\{u_1, ..., u_n\}$ for each possible output.

The maximization problem above is subject to PC & IC:

$$\sum_{i=1}^{n} \pi_{i1} u_i(t_i) - \psi \geq 0 \tag{PC}$$

$$\sum_{i=1}^{n} \pi_{i1} u_i(t_i) - \psi \geq \sum_{i=1}^{n} \pi_{i0} u_i(t_i) \tag{IC}$$

$$\Longrightarrow \sum_{i=1}^{n} (\pi_{i1} - \pi_{i0}) u_i(t_i) \geq \psi$$

(Note: similar to previous notes but with summation operators)

Arguments as in the two output case $\Longrightarrow$ both constraints are binding,

$\Longrightarrow$Lagrangian:

$$L = \sum \pi_{i1} \times [S_i - t_i] \ldots \tag{3}$$
$$+ \mu \left(\sum_{i=1}^{n} \pi_{i1} u_i(t_i) - \psi \right) + \lambda \left(\sum_{i=1}^{n} (\pi_{i1} - \pi_{i0}) u_i(t_i) - \psi \right)$$

$$\lambda \geq 0, \mu \geq 0 \text{ (because binding)} \tag{4}$$

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Taking the FOC's w.r.t. the t's and rearranging yields (as in L&M section 4.4):

$$\frac{1}{u'(t_i)} = \mu + \lambda \frac{\pi_{i1} - \pi_{i0}}{\pi_{i1}} \forall i$$

for $i = 1, 2, 3..n$.

Note that the relative optimal transfers do *not* depend on relative quantities. We assume that, given the output function etc, P has already decided to induce high effort. Now what matters is what each output reveals about A's effort.

Are wages always increasing in output?

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$\Longrightarrow$RHS must increase in i for t to increase in i

I.e., the likelihood ratio of output under different actions must be monotonically increasing with respect to outputs. Basically, the relative probability of high effort must be higher the higher the output.

$\Longrightarrow$the **Monotone Likelihood Ratio Property** (MLRP).

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- But MLRP is a 'rather strong assumption' (L&M)
- Consider a case where effort is 'risky', making spectacularly good and bad outcomes more likely...

Example where MLRP fails:

$$\pi_{10} = \frac{1}{6}, \pi_{20} = \frac{2}{3}, \pi_{30} = \frac{1}{6},$$
$$\pi_{11} = \pi_{21} = \pi_{31} = \frac{1}{3}$$

$$\frac{\pi_{11} - \pi_{10}}{\pi_{11}} = \frac{1}{2}$$
$$\frac{\pi_{21} - \pi_{20}}{\pi_{21}} = -1$$
$$\frac{\pi_{31} - \pi_{30}}{\pi_{31}} = \frac{1}{2}$$

This is not monotone... so MLRP fails.

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- Thus, $t_2 < t_1 = t_3$ to induce $e = 1$(at the lowest cost according to the previous formula). Not monotonic!
- If agent can destroy output secretly, the contract will have to offer the same reward to S1 and S2, leading to a greater necessary risk-premium for the same incentive.

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Information structure for signals:

	$e = 0$	$e = 1$
σ_1	v_{10}	v_{11}
σ_0	v_{00}	v_{01}

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$v_{10} = v_{11}$ (implying $v_{00} = v_{01}$)$\implies$ "The signal is uninformative"

Four states: $y_{11} = \{q_1, \sigma_1\}$, $y_{10} = \{q_1, \sigma_0\}$, $y_{01} = \{q_0, \sigma_1\}$ or $y_{00} = \{q_0, \sigma_0\}$.

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Let π_{jkl}: probabilities of output q_j and signal σ_k given effort $e = l$ (notation distinct from L&M's).

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By independence of the signal and the stochastic component of output:

$$\pi_{jkl} = p(q_j, \sigma_k | e = l) = p(q_j | e = l) \times p(\sigma_k | e = l) = \pi_{jl} v_{kl}$$

"Informativeness principle" / Holstrom's sufficient statistic theorem

Consider transfer t_{jk} for state y_{jk} (output q_j signal σ_k). Previous methods $\Longrightarrow$FOC's:

$$\frac{1}{u'(t_{jk})} = \mu + \lambda \frac{\pi_{jk1} - \pi_{jk0}}{\pi_{jk1}} = \mu + \lambda \frac{\pi_{j1} v_{k1} - \pi_{j0} v_{k0}}{\pi_{j1} v_{k1}} \quad (5)$$

$$v_{k0} = v_{k1} = v_k \;\rightarrow\; \frac{1}{u'(t_{jk})} = \mu + \lambda \frac{\pi_{j1} - \pi_{j0}}{\pi_{j1}}. \quad (6)$$

Note no 'k' on right hand side (7)

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Result (obvious?): P varies wages between states with same output but a different signal if and only if the signal yields some information about the effort chosen.