1 Model: simplified version of Holstrom-Milgrom (1987)

NOTE: Being able to solve the full model is optional, but you should understand it's insights, and be able to define risk aversion and risk premia!

- This is both more general (continuous variables) and more specific (parametrized) than our previous models.
- Effort is now a continuous variable
- Output a continuous (linear) function of effort, chance
- Parametrized (normal) distribution of chance term
- Worker's utility function parametized: "Negative exponential, with Constant Absolute Risk Aversion (CARA)"
- Not separable in income and effort
- No "wealth effects" – incentives, required risk-premium the same no matter the agent's average wealth level.
- This (no wealth effects) will allow P to 'play with' the average pay (PC) and the incentives (IC) independently
- Cost of effort parametrized, nonlinear (increase at an increasing rate)
- This model is not in L&M! (They don't have both effort and output continuous. A similar model is in Varian's Microeconomics text, section 25.4.

1.1 Worker:

Non-discrete effort, nonlinear cost of effort:

$$\begin{aligned} e &\in \mathbb{R}^+ \\ \psi(e) &= \frac{ce^2}{2} \end{aligned}$$

Negative Exponential, Constant Absolute Risk Aversion (CARA) utility:

$$u(t,e) = -\exp^{[-r(t-\psi(e))]}$$

1.2 Firm:

$$\begin{aligned} y &= \alpha + e - \varepsilon \\ & \varepsilon\tilde{\ } N(0, \sigma^2) \end{aligned}$$

Assume principal (firm) is risk neutral, so he will max $E(y - t)$

1.3 Aside: CARA explained

A 'risk premium' (say ρ) to a lottery is defined as the minimum amount someone has to be paid (in addition) to induce them to accept a lottery (say,$\tilde{x}$) rather than the expected value of that lottery (i.e., to make them neutral between the two):

$$E(u(\tilde{x} + \rho)) = u(E(\tilde{x}))$$

We are assuming a worker has a Constant Absolute Risk Aversion (CARA) utility with a risk aversion parameter r.

$$\begin{aligned} u(t, e) &= -\exp^{[-rX]} \\ \text{where } X &= t - \psi(e) \end{aligned}$$

Imagine she faced a 'lottery' $\tilde{x}$ with a probability π of winning wealth a and a probability $1 - \pi$ of winning wealth b.

She would be neutral between this lottery and a 'certainty equivalent' fixed payment of:

$$CE(\tilde{x}) = \pi a + (1 - \pi) b - \rho(\pi, a - b) = E(\tilde{x}) - \rho(\pi, a - b)$$

Where the function $\rho(\pi, a - b)$ is a risk premium; *note it is a function of both the probabilities of each state and the difference in wealth between states; for a* multi-outcome lottery it would be even more complicated.

Define $\Delta A = a - b$ (the difference in outcomes... L&M use 'x')
Given she has the CARA utility function (in wealth x)
$u(x) = -\exp(-rx)$
we can show that this risk premium is:

$$\rho(\pi, \Delta A) = \frac{1}{r} \ln[\pi \exp^{((-r(1-\pi)\Delta A))} + (1 - \pi) \exp^{(r\pi\Delta A)}]$$

This is pretty messy, but we can see:

The risk premium is not a function of *levels* of outcomes (or 'wealth'), only of the *difference* in the outcomes (ΔA) and the *probabilities* of each outcome (defined by π).

$\rho(\pi, \Delta A)$ is increasing in ΔA for all $\Delta A > 0$.

Imagine the payment $\tilde{z}$ is normally distributed around its expected value $E(\tilde{z})$, with variance σ_z^2.

It turns out that in such a case, the risk premium under CARA (with risk aversion parameter r) is:

$$\begin{aligned} \rho(\tilde{z}) &= \frac{r\sigma_z^2}{2} \\ \text{i.e., } CE(\tilde{z}) &= E(\tilde{z}) - \frac{r\sigma_z^2}{2} \end{aligned}$$

Showing $\rho(\pi, \Delta A)$ is increasing in ΔA for all $\Delta A \geq 0, r > 0$ (using maple):

```
> rho:=(1/r)*ln(pi*exp(((-r*(1-pi)*x)))+((1-pi)*exp((r*pi*x))));


> d_rho:=(diff(rho,x));

> assume(pi>0);
> assume(pi<1);
> assume(r>0);
> assume(x>=0);

> sign(d_rho);
 1
```

1.4 Linear Contract

Suppose principal offers linear contract

$$t = \beta_0 + \beta y$$

If worker accepts the contract, will choose e to maximize:

$$\begin{aligned} E(u(e)) &= -\int_{-\infty}^{\infty} \exp^{[-r(t(y(e,\varepsilon))-\psi(e))]} f(\varepsilon)d\varepsilon \\ &= -\int_{-\infty}^{\infty} \exp\{-r[\beta_0 + \beta(\alpha + e + \varepsilon) - \frac{ce^2}{2}]\} f(\varepsilon)d\varepsilon \\ & \text{where } f(\varepsilon) \text{ is the pdf for the normal distribution} \end{aligned}$$

This is a concave problem (for A), so a FOC is sufficient:

$$
\begin{aligned}
\frac{\partial}{\partial e}\int_{-\infty}^{\infty} \exp\{-r[\beta_0+\beta(\alpha+e+\varepsilon)-\frac{ce^2}{2}]\}f(\varepsilon)d\varepsilon &= 0 \Longrightarrow \\
\int_{-\infty}^{\infty} \frac{\partial}{\partial e}\exp\{-r[\beta_0+\beta(\alpha+e+\varepsilon)-\frac{ce^2}{2}]\}f(\varepsilon)d\varepsilon &= 0 \Longrightarrow \\
-r(\beta-ce)\int_{-\infty}^{\infty} \exp\{-r(\beta_0+\beta(\alpha+e+\varepsilon)-\frac{1}{2}ce^2)\}f(\varepsilon)d\varepsilon &= 0
\end{aligned}
$$

Note the term inside the integral is positive. If we set

$$e^* = \frac{\beta}{c}$$

this FOC will hold.

1.5 Case 1: e observable

What is the principal's optimal contract?

When effort observed P can simply maximize the total output minus the amount he must pay A to compensate her for her effort e:

$$S = \alpha + e + \varepsilon - \frac{ce^2}{2}$$

$P's$ problem (simple maximization):

$$\max_e [E(S)] \Longrightarrow e^* = \frac{1}{c}$$

... This will implicitly tell us how to meet the IC constraint when it is needed.

To meet the Participation Constraint:

$$
\begin{aligned}
u(t^*, e^*) &= -\exp^{[-r\{t^*-\psi(e^*)\}]} = \bar{u} = u(\bar{t}) \\
&\quad \text{where } \bar{t} \text{ is her reservation wage (involving 0 effort)} \\
&\Longrightarrow -\exp^{[-r\{t^*-\psi(e^*)\}]} = -\exp^{[-r\{\bar{t}\}]} \\
&\Longrightarrow t^* = \bar{t} + \psi(e^*) \text{ or } t^* = \psi(e^*) \text{ if we set } \bar{t} = 0 \\
\text{(note, if we make this } \bar{t} = 0 \text{ assumption then } \bar{u} &= u(\bar{t}) = -\exp\{0\} = -1)
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow\ & E(S) = \alpha + \frac{1}{c} - \frac{1}{2c} = \alpha + \frac{1}{2c} \\
\text{and } E(y) &= \alpha + \frac{1}{c}
\end{aligned}
$$

1.6 Case 2: e unobservable

Participation constraint:

Certainty equivalent of contract (when $e = e^*$) must be at least $\bar{t}$ (or 0)

Net income (net of effort cost) of agent under contract is random "z"

$$z = \beta_0 + \beta(\alpha + e + \varepsilon) - \frac{ce^2}{2}$$

Note that since z is a linear function of a normally distributed error, it is itself normally distributed.

Under CARA we have:

$$\begin{aligned} CE(z) &= E(z) - r\sigma_z^2 \\ &= [\beta_0 + \beta(\alpha + e) - \frac{ce^2}{2}] - \frac{r\beta^2\sigma^2}{2} \end{aligned}$$

Thus we have the Participation Constraint (if we want *some* effort $\bar{e}$):

$$[\beta_0 + \beta(\alpha + \bar{e}) - \frac{c\bar{e}^2}{2}] - \frac{r\beta^2\sigma^2}{2} \geq \bar{t} \tag{PC}$$

(recall σ^2 is the variance of ε)

And the Incentive Compatability constraint:

$$[\beta_0 + \beta(\alpha + \bar{e}) - \frac{c\bar{e}^2}{2}] - \frac{r\beta^2\sigma^2}{2} \geq \tag{IC}$$

$$[\beta_0 + \beta(\alpha + e) - \frac{ce^2}{2}] - \frac{r\beta^2\sigma^2}{2} \forall e \neq \bar{e} \Rightarrow$$

$$\bar{e} \in \arg\max_e [\beta_0 + \beta(\alpha + e) - \frac{ce^2}{2}] - \frac{r\beta^2\sigma^2}{2}$$

Using arguments as before, we can show that each of these constraints must bind.

Noting this function is concave, to meet the IC we require the FOC

$$\frac{\partial}{\partial e}(\beta_0 + \beta(\alpha + \bar{e}) - \frac{c\bar{e}^2}{2}] - \frac{r\beta^2\sigma^2}{2}) = 0$$
$$\Longrightarrow \beta = c\bar{e}$$

Note this is the same 'pay for effort' formula we solved for earlier.

Note, to get effort e^* we would need $\beta = ce^* = c\frac{1}{c} = 1$. But this might not be optimal *second-best* effort (it isn't).

To meet the IC and the PC exactly we require $\beta = c\bar{e}$ and, (plugging $\beta = c\bar{e}$ from IC into the PC):

$$\beta_0 + c\bar{e}(\alpha + \bar{e}) - \frac{c\bar{e}^2}{2} - \frac{r(c^2\bar{e}^2)\sigma^2}{2} = \bar{t}$$
$$\Rightarrow \beta_0 = \bar{t} - c\bar{e}(\alpha + \bar{e}) + \frac{c\bar{e}^2}{2} + \frac{r(c^2\bar{e}^2)\sigma^2}{2}$$

P's problem (if he 'goes linear'):

$$\begin{aligned} & \max_{e,\beta,\beta_0} E[(\alpha + e + \varepsilon) - \beta_0 - \beta(\alpha + e + \varepsilon)] \\ = & \int_{-\infty}^{\infty} [(\alpha + e + \varepsilon)(1 - \beta) - \beta_0] f(\varepsilon) d\varepsilon \end{aligned} \quad (1)$$

S.t. **IC** and **PC**

But actually, P can choose any compensation scheme, mapping from output to payments.

Holstrom & Milgrom (1987): Linear contracts are optimal anyway.

We assume IC and PC hold with equality, and thus we plug in the requirements on β and β_0 from above into P's problem. This yields the FOC:

(note: $\bar{t}$ will drop out here, as seen below)

$$\frac{\partial}{\partial e} \int_{-\infty}^{\infty} \left((\alpha + e + \varepsilon)(1 - ce) - \left(\bar{t} - ce(\alpha + \frac{e}{2}(1 - cr\sigma^2)) \right) \right) f(\varepsilon) d\varepsilon = 0$$

or

$$\begin{aligned} = \quad & \int_{-\infty}^{\infty} \frac{\partial}{\partial \beta} \left((\alpha + \frac{\beta}{c} + \varepsilon)(1 - \beta) - \left(\bar{t} - \beta(\alpha + \frac{\beta}{2c}(1 - cr\sigma^2)) \right) \right) f(\varepsilon) d\varepsilon \\ = \quad & \int_{-\infty}^{\infty} \left(\frac{1}{c}(1 - \beta) - \varepsilon - \frac{1}{c}\beta - \alpha - \left(\frac{1}{c}\left(cr\sigma^2\beta - \beta \right) - \alpha \right) \right) f(\varepsilon) d\varepsilon \\ = \quad & \left(\frac{1}{c}(1 - \beta) - \frac{1}{c}\beta - \alpha - \left(\frac{1}{c}\left(cr\sigma^2\beta - \beta \right) - \alpha \right) \right) = 0 \\ \Longrightarrow \quad & \beta_{SB} = \frac{1}{cr\sigma^2 + 1} \end{aligned}$$

Thus:

$$\beta_{SB} = \frac{1}{cr\sigma^2 + 1} < 1$$

Remember, to get effort e^* we needed $\beta = ce^* = c\frac{1}{c} = 1$.

So, $\beta_{SB} < 1$, implying the optimal second best contract offers less incentive to output than would be necessary for 'technically efficient' output.

Output is thus 'distorted downward' because the cost of inducing high effort includes paying a risk premium to A.

I.e., the point where the marginal cost (to P) crosses the marginal benefit is lower than with complete information, since costs increase faster.

Comparative statics:

- β_{SB} decreases in r: pay is less incentive-driven when A is more risk-averse
- β_{SB} decreases in c ... when the cost of effort is higher
- β_{SB} decreases in σ^2 ... when the production process is more uncertain
- Note: Simple algebra will yield the base pay β_0^{SB}