1.3 - Moral Hazard, Extension: Continuous Effort, CARA, Linear Contracts

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- This model is not in L&M (they don't have both effort and output continuous)!

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Risk premia and CARA

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Imagine she faced a 'lottery' $\tilde{x}$ with a probability π of winning wealth a and a probability $1-\pi$ of winning wealth b.
She would be neutral between this lottery and a "certainty equivalent" fixed payment of:

$$CE(\tilde{x}) = \pi a + (1-\pi)b - \rho(\pi, a-b) = E(\tilde{x}) - \rho(\pi, a-b)$$

Where the function $\rho(\pi, a-b)$ is a risk premium.

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Given she has the CARA utility function (in wealth x) $u(x) = -\exp(-rx)$ we can show that this risk premium is:

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1. The risk premium is not a function of *levels* of outcomes (or 'wealth'), only of the *difference* in the outcomes (ΔA) and the *probabilities* of each outcome (defined by π). [Discuss]

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2. $\rho(\pi, \Delta A)$ is increasing in ΔA for all $\Delta A > 0$. (See derivation in 2 slides)

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Showing ρ is increasing in ΔA for all $\Delta A \geq 0$, $r > 0$ using maple:

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If worker accepts the contract, will choose e to maximize:

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$$= -\int_{-\infty}^{\infty} \exp\{-r[\beta_0 + \beta(\alpha + e + \varepsilon) - \frac{ce^2}{2}]\} f(\varepsilon) d\varepsilon$$

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This is a concave function, so a FOC is sufficient :

$$\frac{\partial}{\partial e}\int_{-\infty}^{\infty}\exp\{-r[\beta_0+\beta(\alpha+e+\varepsilon)-\frac{ce^2}{2}]\}f(\varepsilon)d\varepsilon=0 \Longrightarrow$$

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Note the term inside the integral is positive for any e. Thus, we must set

$$e^* = \frac{\beta}{c}$$

for the FOC to hold.

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$P's$ problem (simple maximization):

$$\max_e [E(S)] \Longrightarrow e^* = \frac{1}{c}$$

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Under CARA we have:

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And the Incentive Compatibility constraint: [Discuss general constraint, board]

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Using arguments as before, we can show that each of these constraints must bind.

Noting this function is concave, to meet the IC constraint we require the FOC:

$$\frac{\partial}{\partial e}(\beta_0 + \beta(\alpha + \bar{e}) - \frac{c\bar{e}^2}{2}] - \frac{r\beta^2\sigma^2}{2}) = 0$$
$$\implies \beta = c\bar{e}$$

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P's problem – if he 'goes linear':

$$\max_{e,\beta,\beta_0} E[(\alpha + e + \varepsilon) - \beta_0 - \beta(\alpha + e + \varepsilon)]$$
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We assume IC and PC hold with equality, and thus we plug in the requirements on β and β_0 from above. This yields the FOC:

$$\frac{\partial}{\partial e} \int_{-\infty}^{\infty} \left((\alpha + e + \varepsilon)(1 - ce) - \left(\bar{t} - ce(\alpha + \frac{e}{2}(1 - cr\sigma^2)) \right) \right) f(\varepsilon) d\varepsilon = 0$$

or the equivalent FOC:

$$0 = \int_{-\infty}^{\infty} \frac{\partial}{\partial \beta} \left((\alpha + \frac{\beta}{c} + \varepsilon)(1 - \beta) - \left(\bar{t} - \beta(\alpha + \frac{\beta}{2c}(1 - cr\sigma^2)) \right) \right) f(\varepsilon) d\varepsilon$$

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