UNIVERSITY OF ESSEX
DEPARTMENT OF ECONOMICS

Advanced Microeconomics

(EC992-8-AU)

2011/2012

Problem Set 3 (Solutions)

8 A principal employs an agent to work on a project. The worker chooses unobserved effort $e \in \{L, H\}$ at costs c_L and c_H, with $0 < c_L < c_H$. The project succeeds with probability p_H if $e = H$ and p_L if $e = L$, where $1 > p_H > p_L > 0$. The principal pays w_0 if the project fails and w_1 if it succeeds. The agent's utility is given by $u(w_s) - c_e$, where $s \in \{0, 1\}$. Utility $u(\cdot)$ is strictly increasing and strictly concave. The agent has reservation utility $\underline{U}$. The principal's profit is $x_s - w_s$, where x_1 is the output when the project succeeds, and x_0 is the output when the project fails, and $x_1 > x_0$.

(a) Write down the principal's profit maximisation problem.
A: *The principal's profit maximisation problem is*

$$\max_{e,w_0,w_1} \Pi(e, w_0, w_1) = p_e(x_1 - w_1) + (1 - p_e)(x_0 - w_0)$$

subject to the agent's IR constraint

$$p_e u(w_1) + (1 - p_e)u(w_0) - c_e \geq \underline{U}$$

and the agent's IC constraint

$$p_e u(w_1) + (1 - p_e)u(w_0) - c_e \geq p_{e'} u(w_1) + (1 - p_{e'})u(w_0) - c_{e'}$$

for all $e' \in \{L, H\}$.

(b) Suppose the principal chooses to implement $e = L$.

i. Write down the principal's maximisation problem.
A: *The principal's problem in this case is*

$$\max_{w_1,w_0} p_L(x_1 - w_1) + (1 - p_L)(x_0 - w_0)$$

subject to the agent's IR constraint

$$p_L u(w_1) + (1 - p_L)u(w_0) - c_L \geq \underline{U}$$

and the agent's IC constraint

$$p_L u(w_1) + (1 - p_L)u(w_0) - c_L \geq p_H u(w_1) + (1 - p_H)u(w_0) - c_H.$$

ii. What wages will the principal offer in the profit-maximising contract?
A: *Ignore the IC and write down the Lagrangian. Taking the FOC with respect to* w_0 *and then with respect to* w_1*, we obtain*

$$\frac{1}{u'(w_0)} = \frac{1}{u'(w_1)} = \lambda$$

where λ *is the multiplier for the IR constraint. By strict concavity of* u *this implies that* $w_1 = w_0 = w$*. We can now use IR to obtain* $u(w) = c_L + \underline{U}$*. Hence,* $w = u^{-1}(c_L + \underline{U})$*. We now observe that since the wage is constant, IC holds.*

iii. Write down the principal's cost minimisation problem?

A: *The principal's cost minimization problem consist of choosing* w_0 *and* w_1 *that minimise the principal's expected wage payment associated with implementing effort level* e_L*:*

$$\min_{w_1,w_0} p_L w_1 + (1 - p_L) w_0$$

subject to the agent's IR constraint

$$p_L u(w_1) + (1 - p_L) u(w_0) - c_L \geq \underline{U}$$

and the agent's IC constraint

$$p_L u(w_1) + (1 - p_L) u(w_0) - c_L \geq p_H u(w_1) + (1 - p_H) u(w_0) - c_H.)$$

iv. What wages implement $e = L$ efficiently (i.e., in the least costly way for the principal).

A: *This problem is equivalent to that in b.i.. So the solution is the same as the solution of that problem which is obtained in b.ii..*

(c) Now suppose the principal chooses to implement $e = H$.

i. Write down the principal's maximisation problem.

A: *The principal's maximisation problem is*

$$\max_{w_1,w_0} p_H(x_1 - w_1) + (1 - p_H)(x_0 - w_0)$$

subject to the IR constraint

$$p_H u(w_1) + (1 - p_H) u(w_0) - c_H \geq \underline{U}$$

and the IC constraint

$$p_H u(w_1) + (1 - p_H) u(w_0) - c_H \geq p_L u(w_1) + (1 - p_L) u(w_0) - c_L.$$

ii. What wages will the principal offer in the profit-maximising contract?

A: *Write the Lagrangian with* λ *the multiplier for IR and* μ *the multiplier for IC, take the FOC with respect to* w_0 *and* w_1 *and obtain*

$$\frac{1}{u'(w_0)} = \lambda + \mu[1 - \frac{1 - p_L}{1 - p_H}] \tag{1}$$

and

$$\frac{1}{u'(w_1)} = \lambda + \mu[1 - \frac{p_L}{p_H}] \tag{2}$$

Now note that $\mu > 0$*, for otherwise* $w_0 = w_1$ *and therefore IC will be violated. Since the IR and the IC must hold with equality, we can use them to obtain the optimal* w_0 and w_1.

Note: We can also analyze the problem of the optimal wages that implement effort $e = H$ *by analyzing a transformation of the problem stated in b.i.. Instead of choosing* w_1 *and* w_0*, the principal may choose the agent's utility levels* u_1 *and* u_0*. More specifically, let* $\phi(.) = u^{-1}(.)$*. That is,*ϕ is the inverse of u. *The principal's problem can be written as*

$$\max_{u_1,u_0} p_H(x_1 - \phi(u_1)) + (1 - p_H)(x_0 - \phi(u_0))$$

subject to the IR constraint

$$p_H u_1 + (1 - p_H) u_0 - c_H \geq \underline{U}$$

and the IC constraint

$$p_H u_1 + (1 - p_H) u_0 - c_H \geq p_L u_1 + (1 - p_L) u_0 - c_L.$$

The advantage of analyzing this transformed problem is that the constraints are linear. In some cases, this allows us to solve the problem of the optimal contract more easily. Writing the Lagrangian with λ *the multiplier for IR and* μ *the multiplier for IC, take the FOC with respect to* u_0 *and* u_1 *and obtain*

$$\phi'(u_0) = \lambda + \mu[1 - \frac{1 - p_L}{1 - p_H}] \tag{3}$$

and

$$\phi'(u_1) = \lambda + \mu[1 - \frac{p_L}{p_H}]. \tag{4}$$

Note that these conditions are equivalent to (1) and (2), since $u_0 = u(w_0)$, $u_1 = u(w_1)$ *and* $\phi'(.) = 1/u'(.)$ *(this is just the usual rule for the derivative of the inverse of a function, i.e.,* $(f^{-1})' = 1/f'$*). That is,* $\phi'(u_0) = 1/u'(w_0)$ *and* $\phi'(u_1) = 1/u'(w_1)$*. As above, note that* $\mu > 0$*, otherwise* $u_0 = u_1$ *and IC would be violated. Also note that* $\lambda > 0$*. If not, (3) becomes*

$$\phi'(u_0) = \mu[1 - \frac{1-p_L}{1-p_H}] < 0,$$

which is impossible since $\phi' > 0$ *(u is increasing and the inverse of an increasing function is also an increasing function). This means that IR and IC must hold with equality. Hence, we can obtain* u_0 *and* u_1 *directly from them, i.e., the optimal* u_0 *and* u_1 *satisfy*

$$\begin{aligned} p_H u_1 + (1-p_H)u_0 - c_H &= \underline{U} \\ p_H u_1 + (1-p_H)u_0 - c_H &= p_L u_1 + (1-p_L)u_0 - c_L. \end{aligned}$$

Solving this system of equations with respect to u_0 *and* u_1 *we obtain*

$$\begin{aligned} u_0 &= \underline{U} + \frac{c_L p_H - c_H p_L}{p_H - p_L} \\ u_1 &= x + \frac{c_H - c_L - c_H p_L + c_L p_H}{p_H - p_L}. \end{aligned}$$

Hence, the optimal wages are

$$\begin{aligned} w_0 &= \phi(\underline{U} + \frac{c_L p_H - c_H p_L}{p_H - p_L}) = u^{-1}(\underline{U} + \frac{c_L p_H - c_H p_L}{p_H - p_L}) \qquad (5) \\ w_1 &= \phi(\underline{U} + \frac{c_H - c_L - c_H p_L + c_L p_H}{p_H - p_L}) = u^{-1}(\underline{U} + \frac{c_H - c_L - c_H p_L + c_L p_H}{p_H - p_L}). \qquad (6) \end{aligned}$$

iii. Write down the principal's cost minimisation problem.
A: *The principal's cost minimisation problem is*

$$\min_{w_1, w_0} p_H w_1 + (1-p_H)w_0$$

subject to the IR constraint

$$p_H u(w_1) + (1-p_H)u(w_0) - c_H \geq \underline{U}$$

and the IC constraint

$$p_H u(w_1) + (1-p_H)u(w_0) - c_H \geq p_L u(w_1) + (1-p_L)u(w_0) - c_L.$$

iv. What wages implement $e = H$ efficiently (i.e., in the least costly way for the principal).
A: *The principal's cost minimisation problem in c.iii. is equivalent to the principal's maximisation problem in c.i.. So the solution to the two problems is the same.*

v. Show that in the profit-maximising/cost-minimising contracts it holds that $w_1 > w_0$.
A: *We can use (1) and (2) and the fact that* $\mu > 0$ *to obtain this result. Specifically, note that these conditions imply that*

$$\frac{1}{u'(w_0)} < \frac{1}{u'(w_1)}$$

which implies that $u'(w_0) > u'(w_1)$*. This, in turn, implies that* $w_0 < w_1$*, since* u' *is strictly decreasing (recall that u is strictly concave). We can also obtain this same result from (5) and (6). Note that* ϕ *is increasing. Hence we just have to show that*

$$\underline{U} + \frac{c_L p_H - c_H p_L}{p_H - p_L} < \underline{U} + \frac{(c_H - c_L - c_H p_L + c_L p_H)}{p_H - p_L}.$$

This is equivalent to $c_L < c_H$*, which is true by assumption.*

9. Consider the following Principal-Agent model: A firm (the principal) hires a worker (the agent) who is expected to exert effort μ. Effort is unobservable but the firm's revenue $x = \mu + \epsilon$ is publicly observed. The term ϵ is a noise that is distributed $N\left(0, \sigma^2\right)$. The effort is costly to the worker. The cost of effort is given as $c\left(\mu\right) = \frac{k}{2}\mu^2$. The firm offers a linear wage contract $w = \alpha x + \beta$. The firm is risk neutral; the firm's profit is $\pi = x - w$. But the worker is risk averse and has the following CARA utility function $u\left(t\right) = -\frac{1}{r}e^{-rt}$ where $t = w - c\left(\mu\right)$. The worker's reservation utility is $u\left(0\right)$. Note that an artifact of the CARA utility function and normal distribution of ϵ is that the certainty equivalent of the agent for a given μ can be written as: $\alpha\mu + \beta - \frac{k}{2}\mu^2 - \frac{r}{2}\alpha^2\sigma^2$.

(a) Find the optimal values of μ, w, and π for the firm under full information.
A: *The firm's problem under full information is:*

$$\begin{array}{lcr} \max_{\alpha,\beta,\mu} \quad \mathrm{E}_\epsilon \pi = \mathrm{E}_\epsilon\left(x - w\right) = \mathrm{E}_\epsilon\left(x - \left(\alpha x + \beta\right)\right) = \left(\mu - \left(\alpha\mu + \beta\right)\right) & & \\ s.t. \qquad \alpha\mu + \beta - \frac{k}{2}\mu^2 - \frac{r}{2}\alpha^2\sigma^2 = 0 & & (IR) \end{array}$$

One can plug (IR) to eliminate β and the problem boils down to:

$$\max_{\alpha,\mu} \ \mu - \left(\frac{k}{2}\mu^2 + \frac{r}{2}\alpha^2\sigma^2\right)$$

Clearly, this function is decreasing in α. So, at the optimum, $\alpha^{FB} = 0$. The FOC for μ is:

$$1 - k\mu = 0 \text{ or } \mu^{FB} = 1/k.$$

(the superscript FB represents "first best" or full information solution.)
Now, plugging μ^{FB} and α^{FB} back in (IR) we get

$$\beta - \frac{k}{2}\frac{1}{k^2} = 0 \text{ or } \beta^{FB} = \frac{1}{2k}.$$

So, the full information $w^{FB} = \alpha^{FB}x + \beta^{FB} = \frac{1}{2k}$. Finally, the full information $\pi^{FB} = \left(\mu^{FB} + \epsilon\right) - w^{FB} = \left(\frac{1}{k} + \epsilon\right) - \frac{1}{2k} = \frac{1}{2k} + \epsilon$. Hence, $E_\epsilon\pi^{FB} = \frac{1}{2k}$.

(b) For a given wage function (i.e., given α and β) derive the (IC) constraint of the worker.
A: *The (IC) constraint of the worker is*

$$\mu = \arg\max_{\mu'} \alpha\mu + \beta - \frac{k}{2}\mu^2 - \frac{r}{2}\alpha^2\sigma^2$$

The FOC is

$$\alpha - k\mu = 0 \text{ or } \mu = \alpha/k.$$

(c) Since the outside option of the worker is $u\left(0\right)$, the agent's (IR) constraint is $\alpha\mu + \beta - \frac{k}{2}\mu^2 - \frac{r}{2}\alpha^2\sigma^2 = 0$. Write down the principal's problem of choosing wage contract (i.e., values of α and β) and effort level (μ) that maximizes her profit.
A: *The firm's problem is*

$$\begin{array}{lcr} \max_{\alpha,\beta,\mu} & \mathrm{E}_\epsilon\pi = \mu - \left(\alpha\mu + \beta\right) & \\ & & \\ s.t. & \mu = \alpha/k & (IC) \\ & \alpha\mu + \beta - \frac{k}{2}\mu^2 - \frac{r}{2}\alpha^2\sigma^2 = 0 & (IR) \end{array}$$

(d) Show that the optimal $\alpha = \frac{1}{1 + rk\sigma^2}$.
A: *Plugging (IR) and (IC) in the firm's profit function we get,*

$$\max_{\alpha} \ \frac{\alpha}{k} - \frac{\alpha^2}{2k} - \frac{r}{2}\alpha^2\sigma^2$$

So, the FOC is:

$$\frac{1}{k} - \frac{\alpha}{k} - r\alpha\sigma^2 = 0 \text{ or } 1 - \alpha\left(1 + rk\sigma^2\right) = 0$$

or

$$\alpha^* = \frac{1}{1 + rk\sigma^2}.$$

(e) Derive the optimal μ and the associated profit level. Are these quantities smaller or larger than their full information values?

A: *The associated level of* $\mu = \mu^* = \frac{1}{(1+rk\sigma^2)k} = \alpha^* \mu^{FB}$ *and the associated level of profit is*

$$\pi^* = \frac{\alpha^*}{k} - \frac{\alpha^{*2}}{2k} - \frac{r}{2}\alpha^{*2}\sigma^2 = \frac{1}{2k}\left[2\alpha^* - \alpha^{*2}\left(1 + rk\sigma^2\right)\right] = \alpha^* \frac{1}{2k} = \alpha^* \pi^{FB}.$$

Note that both μ^* *and* π^* *are smaller than their full information values as* $\alpha^* < 1$.

(f) How does the optimal α and μ change with σ^2? Give a brief intuition for your finding.

A: *The optimal* μ *and* α *both decreases with* σ^2. *This is the classic risk-incentive trade-off that the firm faces under moral hazard. Higher* σ^2 *means the environment is more noisy and therefore the agent will ask for a higher risk premium whenever his payment is tied to the revenue* (x). *Now, higher is* α *the stronger is the incentive (note that* $w = \alpha x + \beta$*) but the greater is the agent's exposure to risk (and hence, the larger is the risk premium that the firm has to offer). So, when* σ^2 *increases, the firm is better off by lowering* α *(and saving on the risk premium) but that comes at the cost of weaker incentives as reflected by lower* μ.

10. A monopolist decides both the price p and the quality q of the product he sells. Each buyer buys exactly one unit, but buyers vary in terms of their preferences for quality. There are two types of buyers: high-type with utility functions $U^H(q,p) = 2\sqrt{q} - p$ and low-type with utility function $U^L(q,p) = \sqrt{q} - p$. Let the probability of drawing a high-type buyer is 0.2. The cost of production for quality level q is just q (i.e., $c(q) = q$). The monopolist attempts to maximize his profit by offering a menu of contracts $\{(q_L, p_L), (q_H, p_H)\}$. Find the optimal screening contract for the monopolist.

A: *Note that the relevant constraints are as follows*

$$\begin{array}{cc} 2\sqrt{q_H} - p_H \geq 2\sqrt{q_L} - p_L & (IC_H) \\ \sqrt{q_L} - p_L \geq \sqrt{q_H} - p_H & (IC_L) \\ 2\sqrt{q_H} - p_H \geq 0 & (IR_H) \\ \sqrt{q_L} - p_L \geq 0. & (IR_L) \end{array}.$$

Thus, the monopolist problem is to maximize his expected profit

$$\pi = 0.2\left[p_H - q_H\right] + (1 - 0.2)\left[p_L - q_L\right]$$

such that the two (IC) *and the two* (IR) *constraints are satisfied.*

Now (IR_L) *and* (IC_H) *must bind. The argument is as follows: Take* (IR_L) *first. At least one of the two* (IR) *must bind. Else, the monopolist can increase both* p_L *and* p_H. *But the monopolist cannot extract all surplus from the high type. The high type can always say that he is a low type and earn* $2\sqrt{q_L} - p_L > \sqrt{q_L} - p_L \geq 0$. *So, truth-telling must offer him even more. Next, take* (IC_H). *If this does not bind then the monopolist can charge the high-type a bit more. The high-type is still happy with his offer, the low-type has more incentive to tell the truth, and the monopolist makes more money. So, we have*

$$2\sqrt{q_H} - p_H = 2\sqrt{q_L} - p_L$$

and

$$\sqrt{q_L} - p_L = 0.$$

So, $\sqrt{q_L} = p_L$ *and* $\sqrt{q_H} = \left(2\sqrt{q_L} - p_L + p_H\right)/2$. *Or,* $q_L = p_L^2$ *and* $q_H = (p_L + p_H)^2/4$. *Hence, the monopolist's profit maximization problem is*

$$\max_{p_L, p_H} \pi = 0.2\left[p_H - (p_L + p_H)^2/4\right] + 0.8\left[p_L - p_L^2\right]$$

Now, the FOCs are:

$$\frac{\partial \pi}{\partial p_L} = -0.2\frac{(p_L + p_H)}{2} + 0.8\left[1 - 2p_L\right] = 0$$

$$\frac{\partial \pi}{\partial p_H} = 0.2\left[1 - \frac{(p_L + p_H)}{2}\right] = 0$$

So,

$$p_H = \frac{13}{8},\ p_L = \frac{3}{8}$$

Consequently, $q_H = \left(\frac{13}{8} + \frac{3}{8}\right)^2/4 = 1$ *and* $q_L = \left(\frac{3}{8}\right)^2 = 9/64$.